(1a) Let G be a graph on n vertices. Let $\mathcal{S}$ be the set of all (unordered) sets of three distinct vertices in G.
(i) (2 marks) Write an expression for $|\mathcal{S}|$ in terms of n.
(ii) (5 marks) Define $O(.), \Omega(.), \Theta(.)$ notation. (i.e. $f = \Theta(g)$ if $\ldots$).
(iii) (3 marks) Show that $\mathcal{S} = \Theta(n^3)$.

The rest of the question will explore possible values of the influence of boolean functions. We work exclusively with $p = 1/2$ i.e. each of $x \in \mathbb{F}_2^n$ is equally likely to occur.

1b)(i) (2 marks) Define influence of the i-th co-ordinate of f, $I_i^{1/2}(f)$, and total influence $I^{1/2}(f)$.
(ii) (6 marks) Find a boolean function $f : \mathbb{F}_2^n \to \{\text{False}, \text{True}\}$ such that it satisfies

$$\mu_{1/2}(f = \text{False}) = \mu_{1/2}(f = \text{True}) = 1/2 \quad \text{and} \quad I^{1/2}(f) = 1.$$

(You need to give an example of such a function f and prove that it satisfies the two properties above (partial marks will be given for a function which satisfies only one of these properties (with proof).)

1c) (i) (2 marks) Define what it means for $\mathcal{A} \subseteq \mathbb{F}_2^n$ to be a monotone set.
(ii) (8 marks) Let $\mathcal{A} \subseteq \mathbb{F}_2^n$ be monotone and define $f : \mathbb{F}_2^n \to \{-1, 1\}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathcal{A} \\ -1 & \text{if } x \notin \mathcal{A}, \end{cases}$$

then show that $\hat{f}(\{i\}) = -I_i^{1/2}(f)$.

1d) From now on we don't assume f is monotone. Let $f : \mathbb{F}_2^n \to \{-1, 1\}$.
(i) (4 marks) Show that

$$I_i^{1/2}(f) = \frac{1}{2^n} \sum_{x \in \mathbb{F}_2^n} \left(\frac{f(x) - f(x \oplus e_i)}{2} \right)^2.$$

(ii) (4 marks) Define $g_i(x) = f(x) - f(x \oplus e_i)$. For a set $T \in \mathbb{F}_2^n$ write $T \oplus \{i\}$ to mean $T \backslash \{i\}$ if $i \in T$, or $T \cup \{i\}$ if $i \notin T$. Show that

$$\hat{g}_i(S) = \frac{1}{2^n} \sum_{x \,:\, x_i = 0} \Big(f(x) - f(x \oplus e_i) \Big) \Big((-1)^{S \cdot x} - (-1)^{S \cdot (x \oplus e_i)} \Big)$$

(iii) (4 marks) Using (ii) or otherwise, show that

$$\hat{g}_i(S) = \begin{cases} 2\hat{f}(S) & \text{if } i \in S \\ 0 & \text{if } i \notin S, \end{cases}$$

(iv) (5 marks) Using (i) and (iii) or otherwise show that

$$I_i^{1/2}(f) = \sum_{S \,:\, i \in S} \hat{f}(S)^2.$$

(v) (5 marks) Now suppose that $f : \mathbb{F}_2^n \to \{-1, 1\}$ and $\sum_{x \in \mathbb{F}_2^n} f(x) = 0$. Prove

$$I^{1/2}(f) \geq 1.$$

Answers

1a(i) $|\mathcal{S}| = \binom{n}{3}$. (2 marks)

(ii) For functions $f(n), g(n)$ we say that $f(n) = o(g(n))$ if for all $\epsilon > 0$ there exists N_ϵ such that $\forall n > N_\epsilon$: $f(n) < \epsilon g(n)$. (2 marks)
For functions $f(n), g(n)$ we say that $f(n) = O(g(n))$ if there exists constant $C > 0$ and N such that $\forall n > N$: $f(n) < Cg(n)$. (2 marks)
For functions $f(n), g(n)$ we say that $f(n) = \Theta(g(n))$ if there $f(n) = O(g(n))$ and $g(n) = O(f(n))$. (2 marks)

(iii) Observe $|\mathcal{S}| = \frac{n^3}{6}(1 - \frac{3}{n} + \frac{2}{n^2}$. Also, for $n > 12$ the quantities $3/n$ and $2/n^2$ are both less than $1/4$. Hence for $n > 12$ we get that $1/2 < 1 - 3/n + 2/n^2 < 3/2$. This implies for $n > 12$

$$\frac{n^3}{12} < |\mathcal{S}| < \frac{n^3}{4},$$

and so by our definition $|\mathcal{S}| = \Theta(n^3)$. (3 marks)

b(i) Define $I_i^{1/2}(f) = \mu_{1/2}\{x \ : \ f(x) \neq f(x \oplus e_i)\}$ and define $I^{1/2}(f) = \sum_{i=1}^n I_i^{1/2}(f)$. (2 marks)

b(ii) The dictator function $f(x_1, \ldots, x_n)$ defined to be false for $x_1 = 0$ and true for $x_1 = 1$ will be sufficient. Notice $\mu_{1/2}(x \ : \ x_1 = 0) = \mu_{1/2}(x \ : \ x_1 = 1) = 1/2$ so the first condition is satisfied. Now notice the first co-ordinate has influence $I_1^{1/2}(f) = 1$ and for $i \neq 1$ influence is $I_i^{1/2}(f) = 0$ and so the total influence is $I^{1/2}(f) = 1$ as required. (6 marks)

c(i) A set $\mathcal{A} \subseteq \mathbb{F}_2^n$ is monotone if $(x_1, x_2, \ldots x_n) \in \mathcal{A}$ implies that if $y_i \geq x_i$ for each i then $(y_1, y_2, \ldots, y_n) \in \mathcal{A}$ (2 marks).
c(ii) (This is directly from the notes p14) First write out the definition of $\hat{f}(\{i\})$.

$$\hat{f}(\{i\}) = \frac{1}{2^n} \sum_{y \in \mathbb{F}_2^n} f(y)\chi_{\{i\}}(y) = \frac{1}{2^n} \sum_{y \in \mathbb{F}_2^n} f(y)(1_{y_i=0}(y) - 1_{y_1=1}(y)) \tag{1}$$

Notice in (1) the second equality follows by writing out $\chi_{\{i\}}(y)$ in terms of the indicator functions $1_{y_i=0}(y)$ and $1_{y_i=1}(y)$. We can now expand out the sum in (1) to get that

$$\hat{f}(\{i\}) = \frac{1}{2^n} \sum_{y \setminus \{y_i\} \in \mathbb{F}_2^{n-1}} f(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_n\} - f(y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n\}. \tag{2}$$

The equation (2) rearranges nicely. If $f(y) = f(y \oplus e_i)$ then $f(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_n\} - f(y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n\} = 0$ or if $f(y) \neq f(y \oplus e_i)$ then f monotone implies we have $f(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_n\} - f(y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n\} = -2$. The number of times this difference of two will be recorded in (2) is half the number of such y.

$$\hat{f}(\{i\}) = \frac{1}{2^n} \times (-2) \times (|\{y \ : \ f(y) \neq f(y \oplus e_i)\}|/2) = -\frac{1}{2^n}|\{y \ : \ f(y) \neq f(y \oplus e_i)\}| \tag{3}$$

We have now written $\hat{f}(\{i\})$ in terms of the influence of the i-th bit. Notice that (3) calculates the probability of picking a y (under $p = 1/2$) such that $f(y) \neq f(y \oplus e_i)$. Hence,

$$\hat{f}(\{i\}) = -I_i^{\frac{1}{2}}(f). \tag{4}$$

1d(i)

$$\begin{aligned}
I_i^{1/2}(f) &= \frac{1}{2^n}\Big|\{x \in \mathbb{F}_2^n \ : \ f(x) \neq f(x \oplus e_i)\}\Big| \\
&= \frac{1}{2^n}\sum_x 1_{\{f(x)\neq f(x\oplus e_i)\}}(x) \\
&= \frac{1}{2^n}\sum_x \left(\frac{f(x) - f(x \oplus e_i)}{2}\right)^2
\end{aligned}$$

d(ii) By defintion of $g_i(x)$ we get

$$\hat{g}_i(S) = \frac{1}{2^n}\sum_T \big(f(T) - f(T \oplus i)\big)(-1)^{|S\cap T|}.$$

Now seperate the sum over those T which contain i and those T which do not and then re-arrange

$$\begin{aligned}
\hat{g}_i(S) &= \frac{1}{2^n}\Big(\sum_{T:i\notin T}\big(f(T) - f(T\cup\{i\})\big)(-1)^{|S\cap T|} + \sum_{T:i\in T}\big(f(T) - f(T\backslash\{i\})\big)(-1)^{|S\cap T|}\Big) \\
&= \frac{1}{2^n}\Big(\sum_{T:i\notin T}\big(f(T) - f(T\cup\{i\})\big)(-1)^{|S\cap T|} + \big(f(T\cup\{i\}) - f(T)\big)(-1)^{|S\cap(T\cup\{i\}|}\Big) \\
&= \frac{1}{2^n}\Big(\sum_{T:i\notin T}\big(f(T) - f(T\cup\{i\})\big)\Big((-1)^{|S\cap T|} - (-1)^{|S\cap(T\cup\{i\}|}\Big).
\end{aligned}$$

then defining x such that $x_i = 1$ if $i \in T$ we get the required expression. (4 marks).
d(iii) First case if $i \in S$ then

$$\begin{aligned}
\hat{g}_i(S) &= \frac{1}{2^n}\Big(\sum_{T:i\notin T}\big(f(T) - f(T\cup\{i\})\big)\Big((-1)^{|S\cap T|} + (-1)^{|S\cap T|}\Big). \\
&= \frac{1}{2^n}\Big(2\sum_{T:i\notin T} f(T)(-1)^{|S\cap T|} - 2\sum_{T:i\in T} f(T\cup\{i\})(-1)^{|S\cap T|}(-1)\Big). \\
&= \frac{1}{2^n}\Big(2\sum_T f(T)(-1)^{|S\cap T|} \\
&= 2\hat{f}(S).
\end{aligned}$$

If $i \notin S$ then

$$\begin{aligned}\hat{g}_i(S) &= \frac{1}{2^n}\Big(\sum_{T: i \notin T} \big(f(T) - f(T \cup \{i\})\big)\Big((-1)^{|S \cap T|} - (-1)^{|S \cap T|}\Big). \\ &= 0.\end{aligned}$$

(4 marks)

d(iv) From (i) and Parseval

$$\begin{aligned}I_i^{1/2}(f) &= \frac{1}{2^n}\frac{1}{4}\sum_x (f(x) - f(x \oplus e_i))^2 \\ &= \frac{1}{2^n}\frac{1}{4}\sum_x g_i(x)^2 \\ &= \frac{1}{2^n}\frac{1}{4}\sum_S \hat{g}_i(S)^2\end{aligned}$$

Split the above sum over S into those sets S which contain i and those which don't and then apply (iii):

$$\begin{aligned}I_i^{1/2}(f) &= \frac{1}{2^n}\frac{1}{4}\Big(\sum_{S: i \in S} \hat{g}_i(S)^2 + \sum_{S: i \notin S} \hat{g}_i(S)^2\Big) \\ &= \frac{1}{2^n}\frac{1}{4}\Big(\sum_{S: i \in S} 4\hat{f}(S)^2 + 0\Big) \\ &= \frac{1}{2^n}\sum_{S: i \in S} \hat{f}(S)^2.\end{aligned}$$

(5 marks)