Exercise 1. An *Eulerian circuit* of G is a sequence of vertices $v_1v_2\dots v_\ell$ (a vertex may appear more than once) so that every edge $uw \in E(G)$ appears as v_iv_{i+1} for some i in the sequence, and so that $v_1 = v_\ell$. A Eulerian graph is one which has a Eulerian circuit.

A *Hamiltonian cycle* of graph G on atleast three vertices is an sequence $v_1v_2\dots v_n$ such that each $u \in V(G)$ appears exactly once, $v_1 = v_2$ and each $v_iv_{i+1} \in E(G)$. A graph is Hamiltonian if it has a Hamiltonian cycle.

(a) Let $\mathcal{A}$ be the set of Eulerian graphs. Show that $\mathcal{A}$ is not monotone.

(b) Let $\mathcal{B}$ be the set of Hamiltonian graphs. Is $\mathcal{B}$ monotone?

Exercise 2. A graph G with $n \geq 3$ vertices, denoted C_n, is a cycle if its vertices can be (re)-labelled $v_1,\dots,v_n$ such that $E(G) = \{v_iv_{i+1} : i \in [n]\}$ where the subscript addition is taken modulo n. For example a cycle on 3 vertices is  and there are three cycles on four vertices , , .

A *connected graph* is one in which any two vertices uv are connected by a sequence of vertices $v_1\dots v_\ell$ so that $u = v_1$, $v = v_\ell$ and each v_iv_{i+1} is an edge. For example  is connected but  is not connected.

(a) A graph with n vertices and n edges must contain a cycle as a subgraph.

(b) A connected graph with n vertices and n edges must contain exactly one cycle.

(c) Give an example to show that the assumption of connectivity is needed for part b.

Exercise 3. (Covered in lectures) Let $\mathcal{A}_\triangle$ be the set of all graphs which contain  as a subgraph.

(a) Show that $\mathbb{P}(G(n,1/2) \in \mathcal{A}_\triangle) \to 1$.

(b) (optional) Fix a constant $0 < p \leq 1$, and show that $\mathbb{P}(G(n,p) \in \mathcal{A}_\triangle) \to 1$.

Exercise 4. (Covered in lectures) Prove the following:

Let $X_1, X_2, \dots$ be a sequence of random variables each taking non-negative values. If $\mathbb{E}[X_n] \to 0$ then

$$\mathbb{P}(X_n = 0) \to 1,$$

and if $\mathbb{E}[X_n] > 0$ for each n, and $\mathbb{V}[X_n]/\mathbb{E}[X_n] \to 0$ then

$$\mathbb{P}(X_n = 0) \to 0.$$

Exercise 5. Show whp $np \to \infty$ implies whp G_n contains  i.e. a 3-cycle[1].

Let Y_n count the number of  in G_n and for any 3-subset of vertices $S \subset V(G)$ let A_S be the event that G_n restricted to the vertices S is a .

(a) Show by linearity of expectation that:

$$\mathbb{V}(Y_n) = \sum_{S,T \in \binom{[n]}{3}} \Big(\mathbb{P}(A_S \,\&\, 1_{A_T}) - \mathbb{P}(A_S)\mathbb{P}(A_T) \Big).$$

[1]This exercise demonstrates a different way to prove the second part of Theorem **??**. In the proof we showed that whp $e(G_n) \geq n$ for $np \to \infty$ and from this and Q 2a we concluded that $np \to \infty$ implies whp G_n has a cycle.

(b) Notice that when the sets of vertices S and T don't intersect that the events A_S and A_T are independent. What about when they intersect on one vertex? Using (a) show that:

$$\mathbb{V}(Y_n) \leq \sum_{|S \cap T| = \{2,3\}} \mathbb{P}(A_S \,\&\, A_T).$$

(c) After some case analysis and from (b) show: $\mathbb{V}(Y_n) \leq n^4 p^5 + n^3 p^3$.

(d) From (c) conclude that whp $Y_n > 0$. *Hint: use Chebyshev's inequality.*

Exercise 6. Given $k \in \mathbb{N}$, let $\mathcal{P}_k$ be the set of graphs which have a path on k vertices as a subgraph.

(a) (Covered in lectures) Find the threshold function for $\mathcal{P}_3$ (notice $\mathcal{P}_3$ is the set of graphs containing the path  as a subgraph).

(b) Find the threshold for $\mathcal{P}_4$.

(c) (optional) Let $k \in \mathbb{N}$ be a constant. Find the threshold for $\mathcal{P}_k$ in terms of k and n.

Exercise 7. For each of the following boolean functions f, aka voting schemes, find a set S such that the function is expressible in terms of that character, i.e. $f(x) = \chi_S(x)$ or $f(x) = -\chi_S(x)$.

(a) The dictator function, $Dict_n^1(x) = x_1$.

(b) The parity function, $Par(x)$.

(c) The XOR function of the first two inputs, $f(x) = XOR(x_1, x_2)$.

(d) The constant function $f(x) = 1$.

Exercise 8. We can define an interated majority function for $n = 3^k$. The base case is $\text{Imaj}_1(x_1, x_2, x_3) = \text{Maj}_3(x_1, x_2, x_3)$ and

$$\text{Imaj}_k(x) = \text{Maj}_3(\text{Imaj}_{k-1}(x_1, \ldots, x_{3^{k-1}}), \text{Imaj}_{k-1}(x_{3^{k-1}+1}, \ldots, x_{2.3^{k-1}}), \text{IMaj}_{k-1}(x_{2.3^{k-1}+1}, \ldots, x_{3^k})).$$

For example, for $k = 2$, $\text{Imaj}_2(x_1, \ldots, x_9) = \text{Maj}_3(\text{Maj}_3(x_1, x_2, x_3), \text{Maj}_3(x_4, x_5, x_6), \text{Maj}_3(x_7, x_8, x_9))$.

(a) Calculate the influence of the i-th bit $I_i(\text{Imaj}_2)$ and total influence $I^p(\text{Imaj}_2)$.

(b) Can you calculate $I_i^p(\text{Imaj}_k)$ and $I^p(\text{Imaj}_k)$? You may take $p = 1/2$ if you like.