# Modularity and Graph Expansion

Fiona Skerman

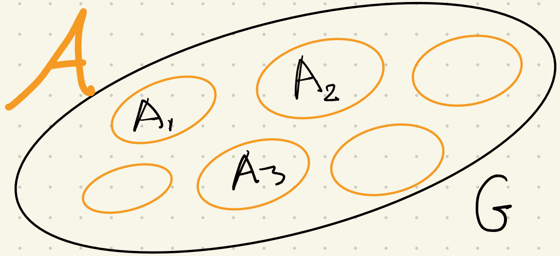
with Baptiste Louf + Colin McDiarmid

arXiv: 2312.07521

# Modularity

'meas. of how well a graph can be clustered'

NEWMAN + GIRVAN 2004.




graph G, m edges. $\mathcal{A} = \{A_1, \dots, A_k\}$ vertex partition

score of partition $\mathcal{A}$, $q_{\mathcal{A}}(G) =$

modularity of G $\quad q^*(G) = \max_{\mathcal{A}} q_{\mathcal{A}}(G)$

"high vals taken to indicate more community structure"

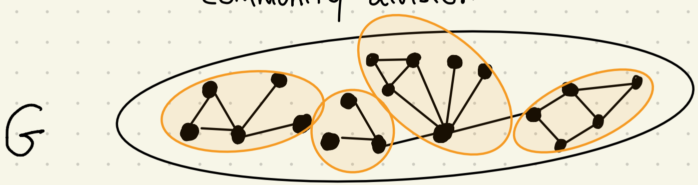
# Community Detection

input graph $G = (V, E)$

vertices nodes — edges (weighted)

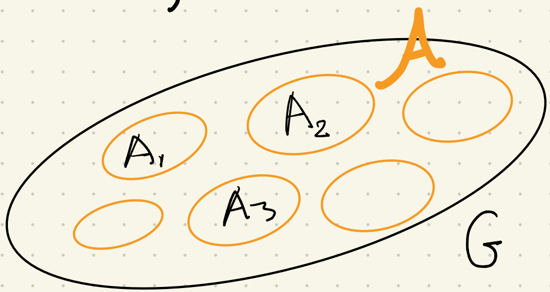
output vertex partition $\mathcal{A}$

'community division'




- modularity score NP-hard to opt.
- Louvain ~ modularity based
  ~ iteratively build a partition local choices – maximise mod.
- most popular methods use modularity

# Modularity 'meas. of how well a graph can be clustered'

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$\mathrm{vol}(A) = \sum_{u \in A} d_u$

$$q_{\mathcal{A}}(G) = \sum_{A \in \mathcal{A}} \frac{e(A)}{m} - \frac{\mathrm{vol}(A)^2}{(2m)^2} = \frac{1}{2m} \sum_{A \in \mathcal{A}} \sum_{u,v \in A} \mathbb{1}[u \sim v] - \frac{d_u \cdot d_v}{2m}$$

"edge contrib" "degree tax"

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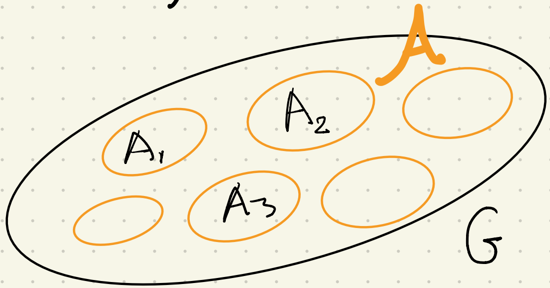


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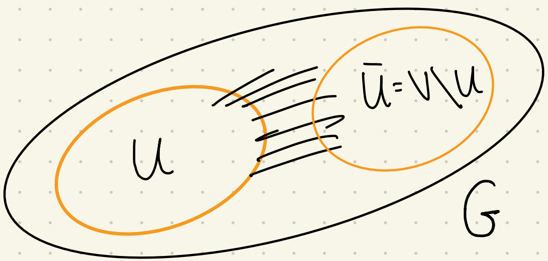
# Expansion




conductance

$$h_G = \min_{U \subseteq V(G)} \frac{e(U, \bar{U})}{\min\{vol(U), vol(\bar{U})\}}$$

$$\hat{h}_G = \min_{U \subseteq V(G)} \frac{e(U, \bar{U})\, vol(G)}{vol(U)\, vol(\bar{U})}$$

$$h_G \leq \hat{h}_G \leq 2 h_G$$

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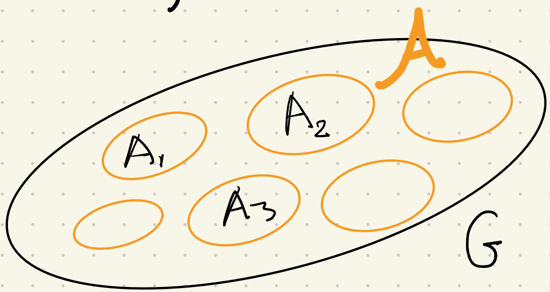


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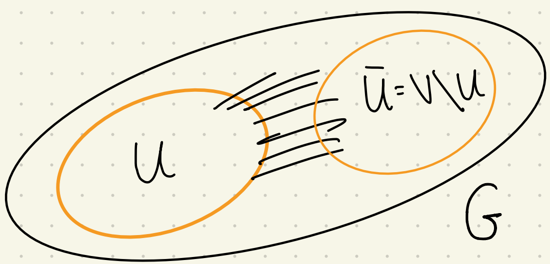


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# Robustness

$$|q^*(G + 1) - q^*(G)| < \frac{2}{e(G)}$$

but $\qquad h_{G+1} = \hat{h}_{G+1} = 0 \qquad$ disconnected!

# Modularity

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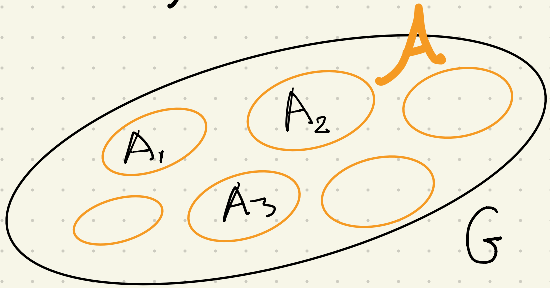


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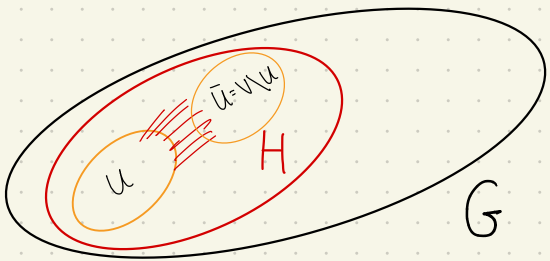
# Expansion of Subgraphs

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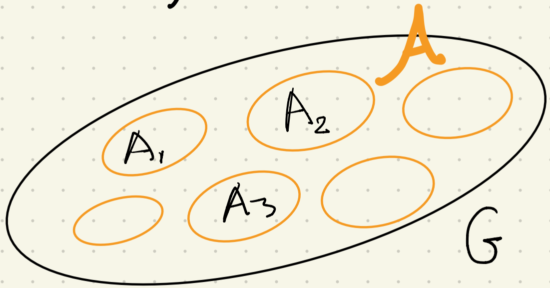


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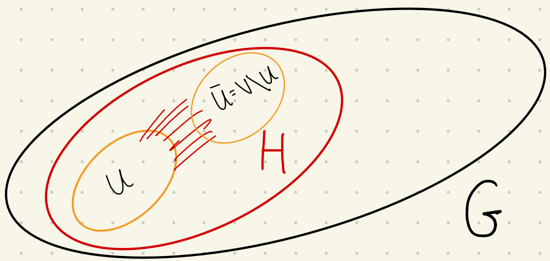


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**Thm** (informal)

$G$ has $q^*(G) \sim 1$

<=>

$G$ has no large expander subgraphs

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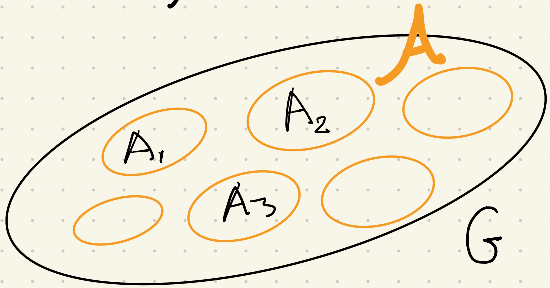


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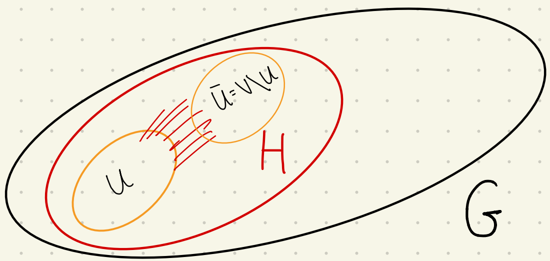


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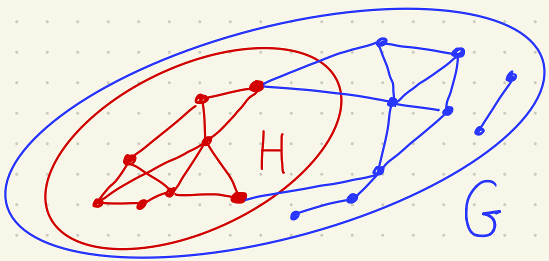


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**Thm** $\forall\, 0 < \alpha < 1$, $\forall \varepsilon > 0$,

(a) $G$ has subgraph $H$, $\frac{e(H)}{e(G)} \geqslant \alpha$, $h_H \geqslant \alpha$

$\Rightarrow q^*(G) \leq 1 - \alpha^2$

(b) $\exists \delta > 0$: $q^*(G) \leq 1 - f(\alpha) - \varepsilon$

$\Rightarrow G$ has induced subgraph $H$,

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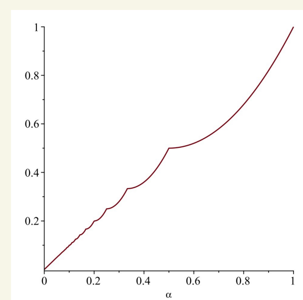
$f(\alpha) := \max\{\sum_i x_i^2 : 0 \leq x_i \leq \alpha, \sum_i x_i = 1\}$

NB. $f(\alpha) \approx \alpha$ for small $\alpha$.

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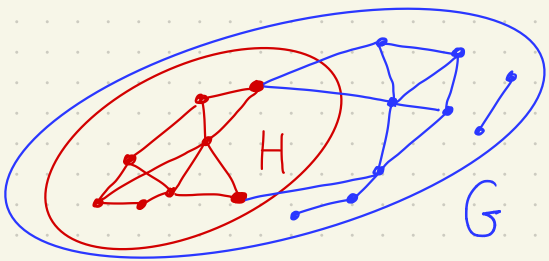


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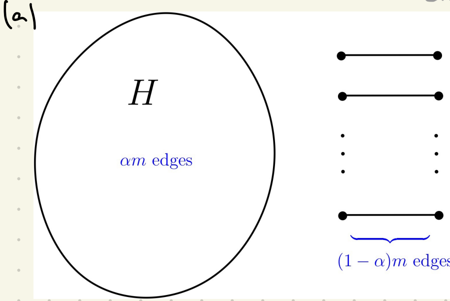
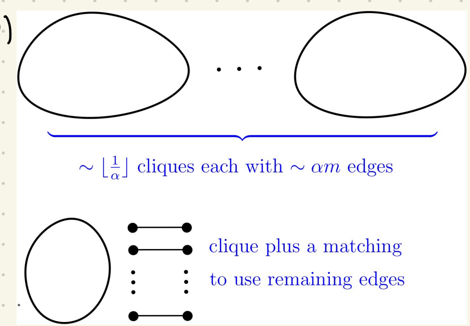



$q^*(G_H) > 1 - \alpha^2 - \varepsilon$

(b)




$q^*(G_\alpha) < 1 - f(\alpha) + \varepsilon$

any $H$, with $\frac{e(H)}{e(G)} \geq \alpha$ is disconnected

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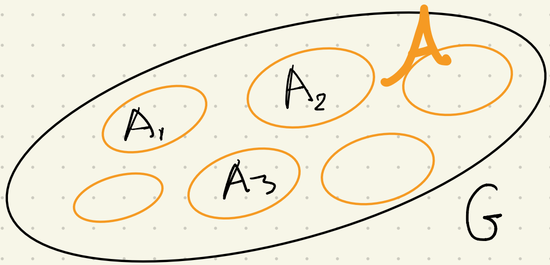


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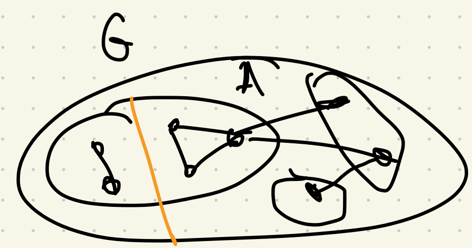
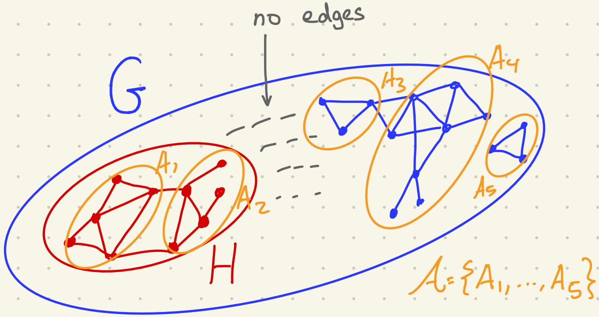
**Obs**

- $\mathcal{A}$ optimal partition of $G$ i.e $q_{\mathcal{A}}(G) = q^*(G)$
- $\forall A \in \mathcal{A}$ $\quad G[A]$ conn. $\quad$ (+ isolated vert)



# Modularity: Resolution Limit



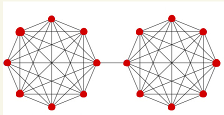

OPT(G) : set of partitions achieving maximal modularity

FORTUNATO + BARTHÉLEMY 2008.

## Res Limit

H is a connected component in graph G.

- $\forall H$ : $e(H) < \sqrt{2\, e(G)} \Rightarrow$ H not split
- H 'dumbell graph' $e(H) > \sqrt{2\, e(G)} \Rightarrow$ H split



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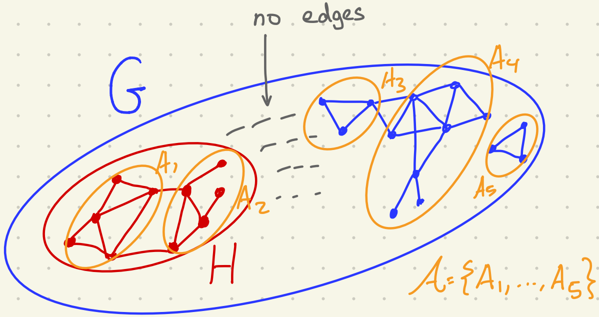


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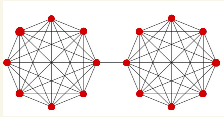

$\mathrm{OPT}(G)$ : set of partitions achieving maximal modularity

**Thm** $H$ is a connected component in graph $G$. Let

$$\alpha = \frac{e(H)}{e(G)} \qquad \hat{h}_H = \min_{U \subseteq V(H)} \frac{e(U,\bar{U})\,\mathrm{vol}(H)}{\mathrm{vol}(U)\,\mathrm{vol}(\bar{U})}.$$

Then

$\alpha > \hat{h}_H \Rightarrow \forall \mathcal{A} \in \mathrm{OPT}(G)$ $H$ split

$\alpha < \hat{h}_H \Rightarrow \forall \mathcal{A} \in \mathrm{OPT}(G)$ $H$ not split

$\alpha = \hat{h}_H \Rightarrow \exists \mathcal{A}, \mathcal{A}' \in \mathrm{OPT}(G)$

in $\mathcal{A}$ $H$ split

in $\mathcal{A}'$ $H$ not split

**Open**

## Upper Bound for Modularity in terms of conductance $h_G$

**Cor** $\forall G: \quad q^*(G) \leq 1 - \min\{\hat{h}_G, 1\} \leq 1 - h_G$

Tight for $\hat{h}_G$: $\forall \hat{\delta} \quad 0 \leq \hat{\delta} \leq 1 \quad \forall \varepsilon > 0 \quad \exists G$ s.t.

$|\hat{h}_G - \hat{\delta}| < \varepsilon \qquad q^*(G) > 1 - \hat{h}_G - \varepsilon$

**Construction**

$G$ $k$-clique, $\ell \geq 1$ leaves @ each $v$

$\hat{h}_G = \dfrac{k}{2\ell + k - 1} \qquad q^*(G) \geq 1 - \hat{h}_G - o_k(1)$

**Open**

### What is the optimal $f$ s.t $\forall G: \quad q^*(G) \leq 1 - f(h_G)$ ?

By  construction $\quad x \leq f(x) \leq 2x$

$q^*(G) \geq 1 - 2h_G - o_k(1)$

**Recall**

$$h_G = \min_{U \subseteq V(G)} \frac{e(U, \bar{U})}{\min\{vol(U), vol(\bar{U})\}}$$

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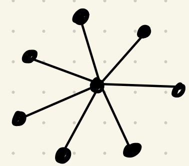
$\delta = 1$



$q^* = 0$

$\Rightarrow f(1) = 1$

$\delta \leq \varepsilon$

$q^* \geq 1 - \varepsilon$

$\Rightarrow f(0) = 0$