Cellular Automata I

Modelling Complex Systems

Some of this lecture is adapted from:
Lectures of Irene Marcovici and
previous slides of David Sumpter.

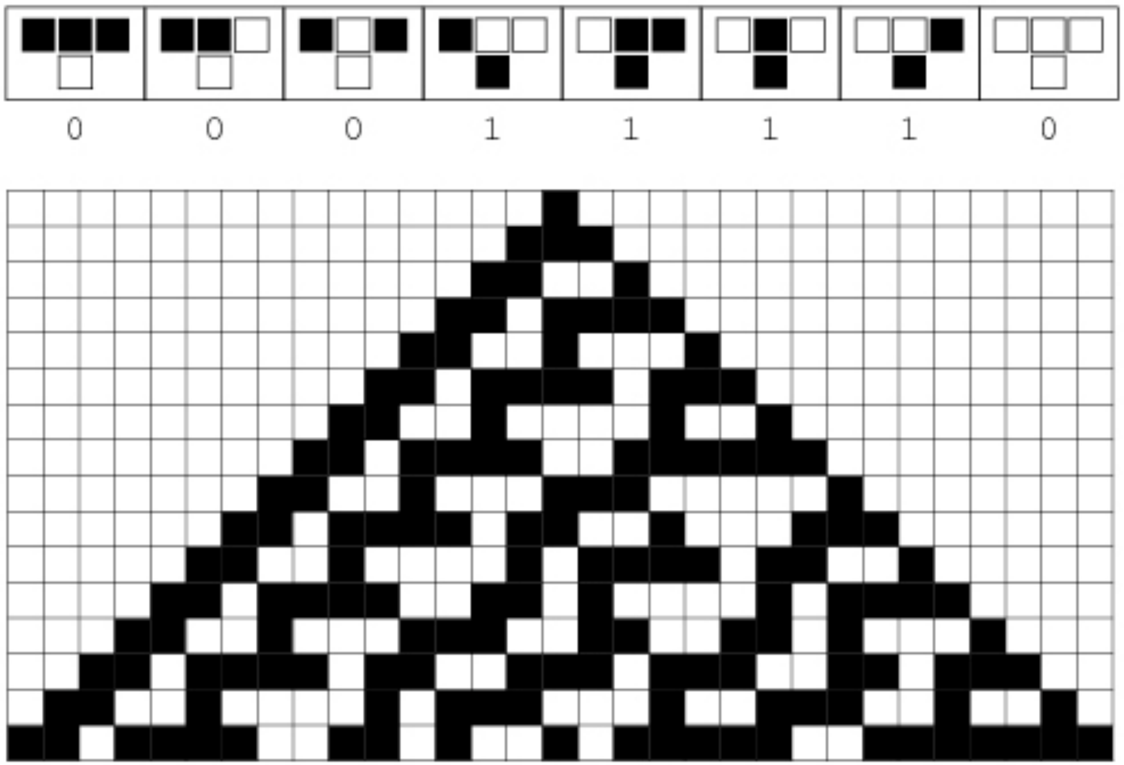
What is a cellular automata (CA)?

A CA consists of an array of cells each with an integer “state”.

On each time step a local update rule is applied to the cells. The update rule defines how the particular cell will update its state as a function of its neighbours state.

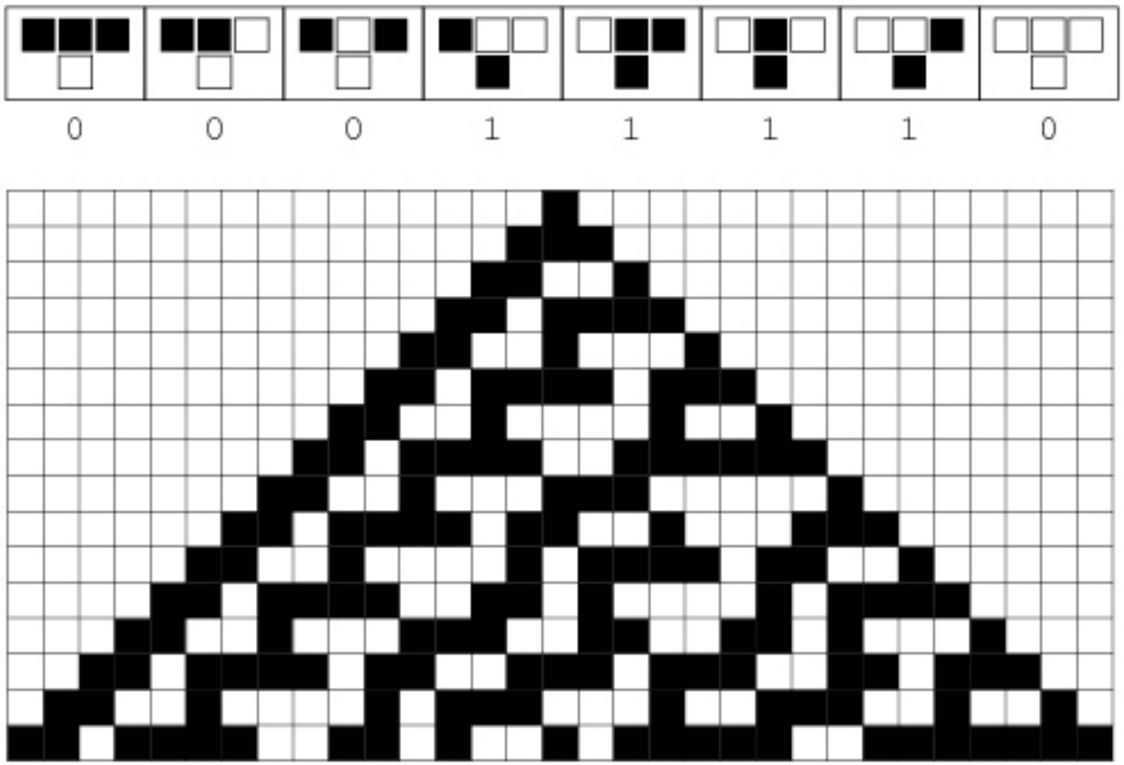
The CA is run over time and the evolution of the state is observed.

elementary cA



- white = 0, black = 1
- 111 —> 0
  110 —> 0
  101 —> 0
  100 —> 1
  011 —> 1
  010 —> 1
  001 —> 1
  000 —> 0

elementary cA



- 2^8 = 256 rules in total
- rule 0 — rule 255

elementary cA

- Classified based on patterns
- Class 1: **<u>Fixed</u>**; all cells converge to a constant 0 or 1 set

  Class 2: **<u>Periodic</u>**; repeats the same pattern, like a loop

  Class 3: **<u>Chaotic</u>**; pseudo-random

  Class 4: **<u>Complex local structures</u>**; exhibits behaviours of both class 2 and class 3; with long lived hard to classify structure
- Feels that we understand class 1 - 3, but not 4.

elementary cA

- Class 1: **Fixed**; e.g., rule 8 (00001000)
- Class 2: **Periodic**; e.g., rule 50 (00110010)
- Class 3: **Chaotic**; e.g., rule 30 (00011110)
- Class 4: **Complex local structures**; e.g., rule 110 (01101110)

elementary cA

- Class 1: <u>**Fixed**</u>; e.g., rule 8 (00001000)
- Class 2: <u>**Periodic**</u>; e.g., rule 50 (00110010)
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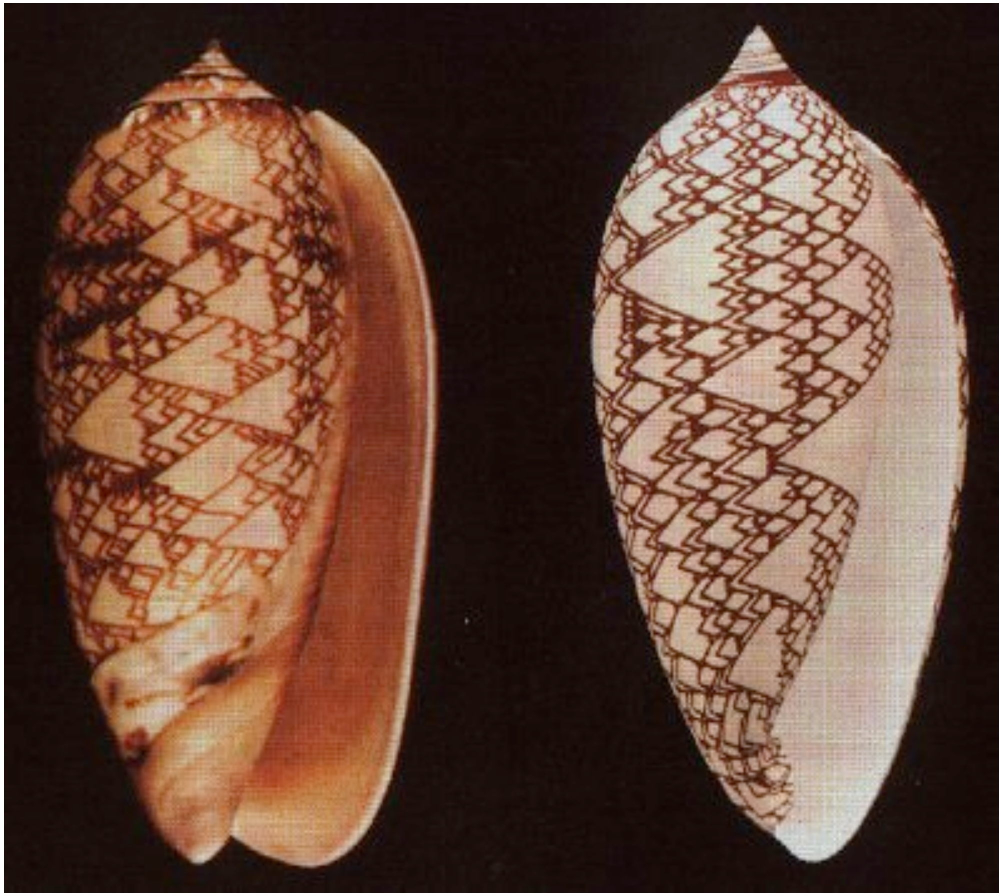
elementary cA



More complex cA

- CA can be extended:

1. More states for single grid
2. Longer range interactions
3. Two or more dimensions
4. Hexagonal or other grids

……

Formal definition

- We start from a **configuration**.
- All the cells update **simultaneously** their colour, and choose their new colour in function of the colours they observe in a **finite neighbourhood**.

If all cells apply simultaneously the same local rule, the update dynamics is called a **cellular automaton**.

Formal definition: cellular automata on inf line

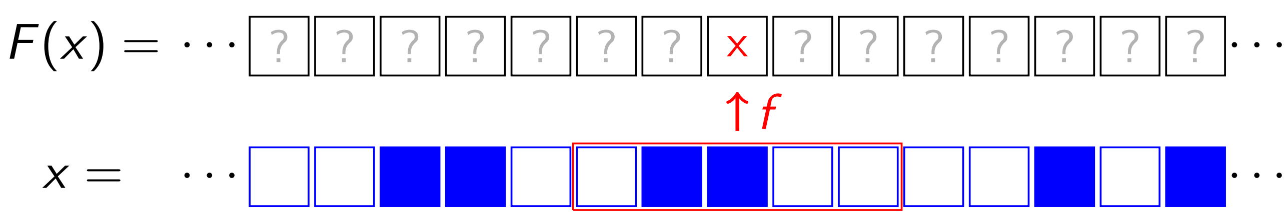
Let $\mathcal{A}$ be a finite set of symbols, called the **alphabet**.

We denote by $\mathcal{A}^{\mathbb{Z}}$ the set of **configurations**.
An element of $\mathcal{A}^{\mathbb{Z}}$ is a sequence $(x_k)_{k\in\mathbb{Z}}$, with $x_k \in \mathcal{A}$ for $k \in \mathbb{Z}$.

Definition

A map $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a **cellular automaton** if there exists a **radius** $r \geq 0$ and a **local function** $f : \mathcal{A}^{2r+1} \to \mathcal{A}$ such that:

$$F(x)_k = f(x_{k-r}, \ldots, x_{k+r-1}, x_{k+r}).$$



$\mathcal{A} = \{\square, \blacksquare\}$, $r = 2$

Irene Marcovici

Game of life

- **World**: 2D orthogonal grid of square cells
- **States**: Dead (0, white) or Alive (1, black)
  - Reproduction: 0 -> 1, if #(Alive neighbours) = 3
  - Surviving: 1 -> 1, if #(Alive neighbours) = 2 or 3
  - Underpopulation: 1 -> 0, if #(Alive neighbours) < 2
  - Overpopulation: 1 -> 0, if #(Alive neighbours) > 3
  - Otherwise, no change

Game of life

- Reproduction:

0 -> 1, if #(Alive neighbours) = 3

- Surviving:

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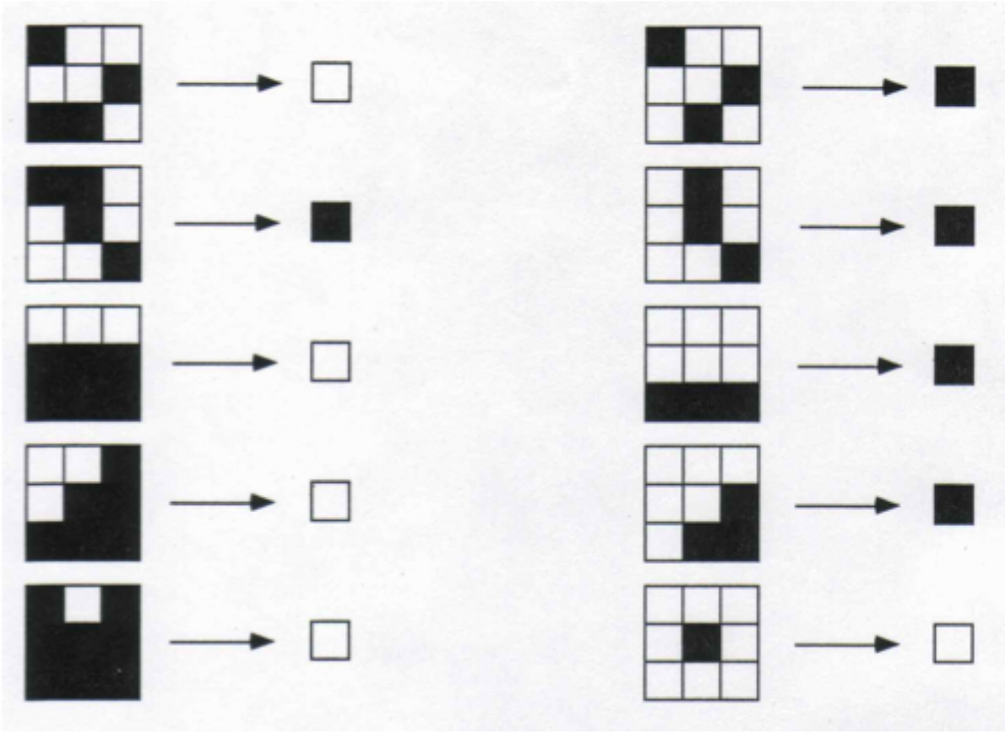
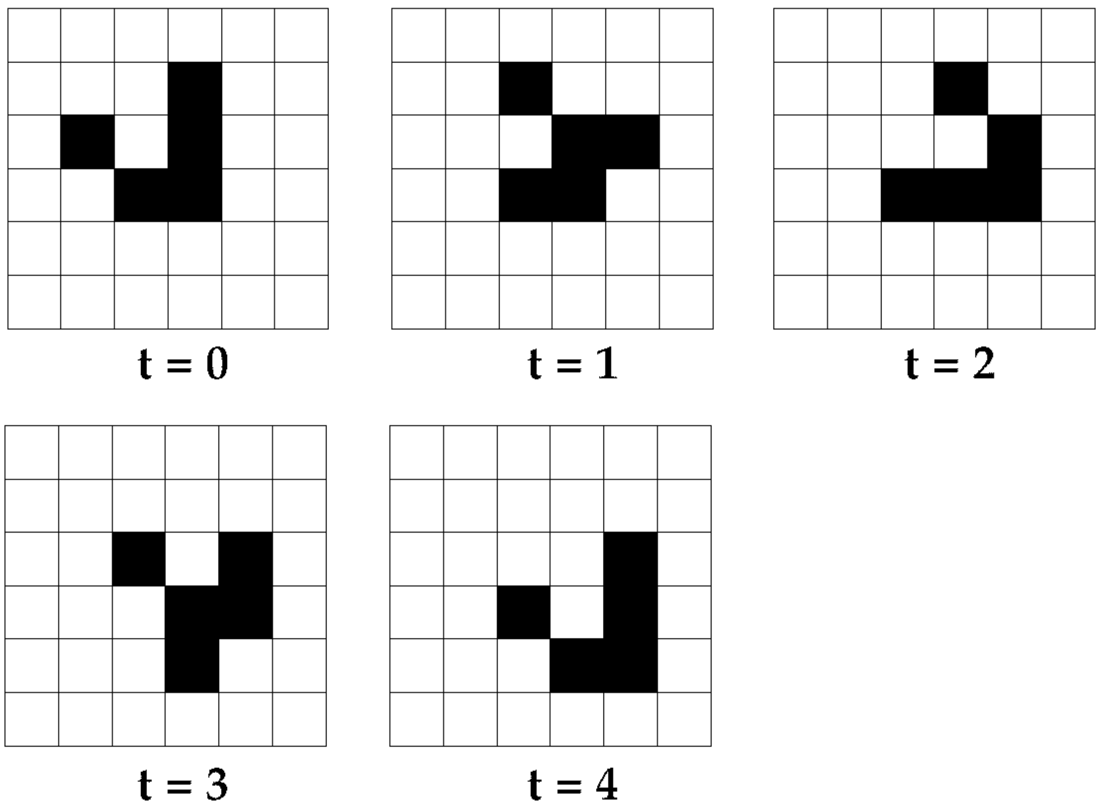


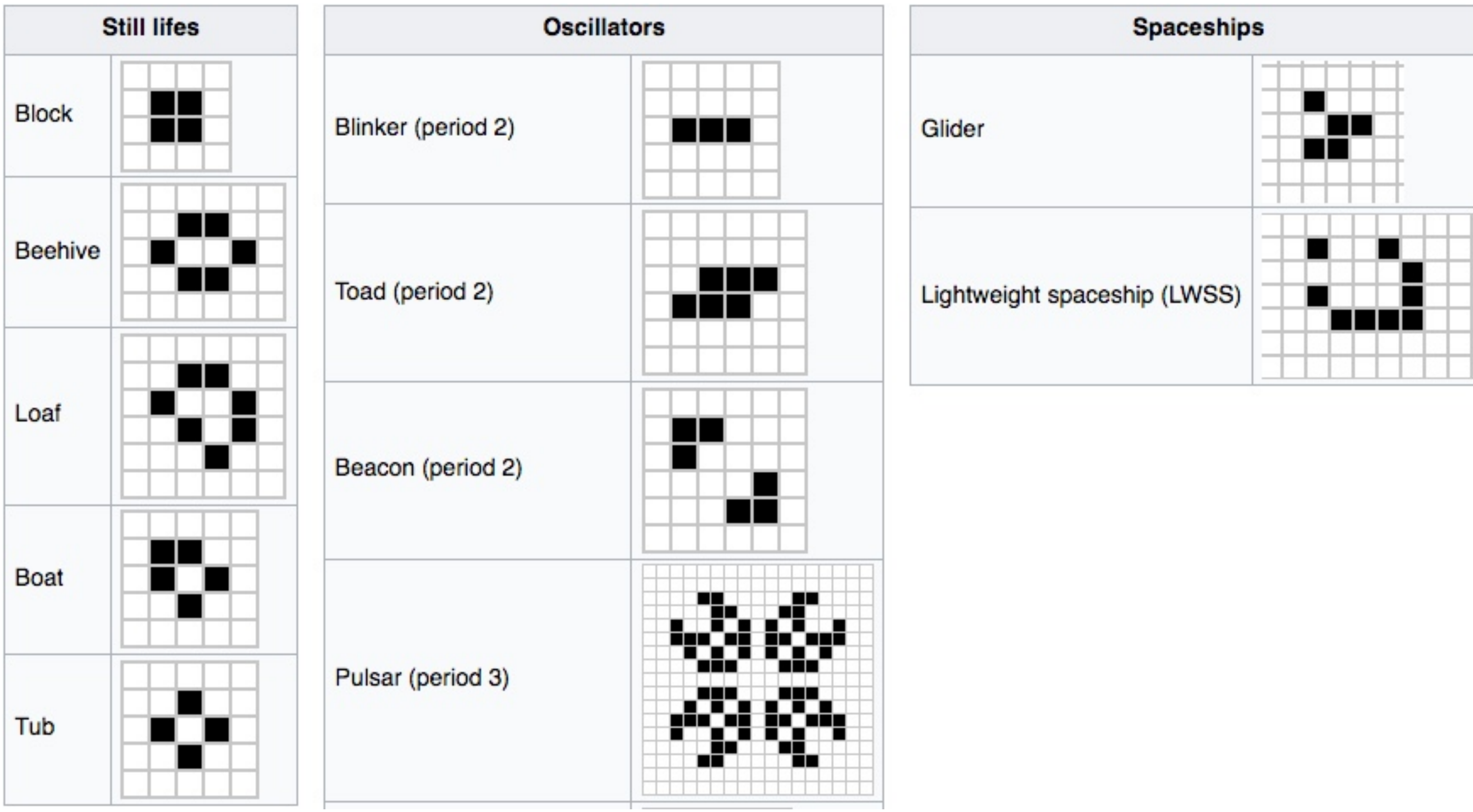
Game of life

- Example: Glider



Game of life

- More examples:



https://en.wikipedia.org/wiki/Conway's_Game_

Large-scale structures

https://vimeo.com/5428232

Computational gates

https://www.youtube.com/watch?v=vGWGeund3eA

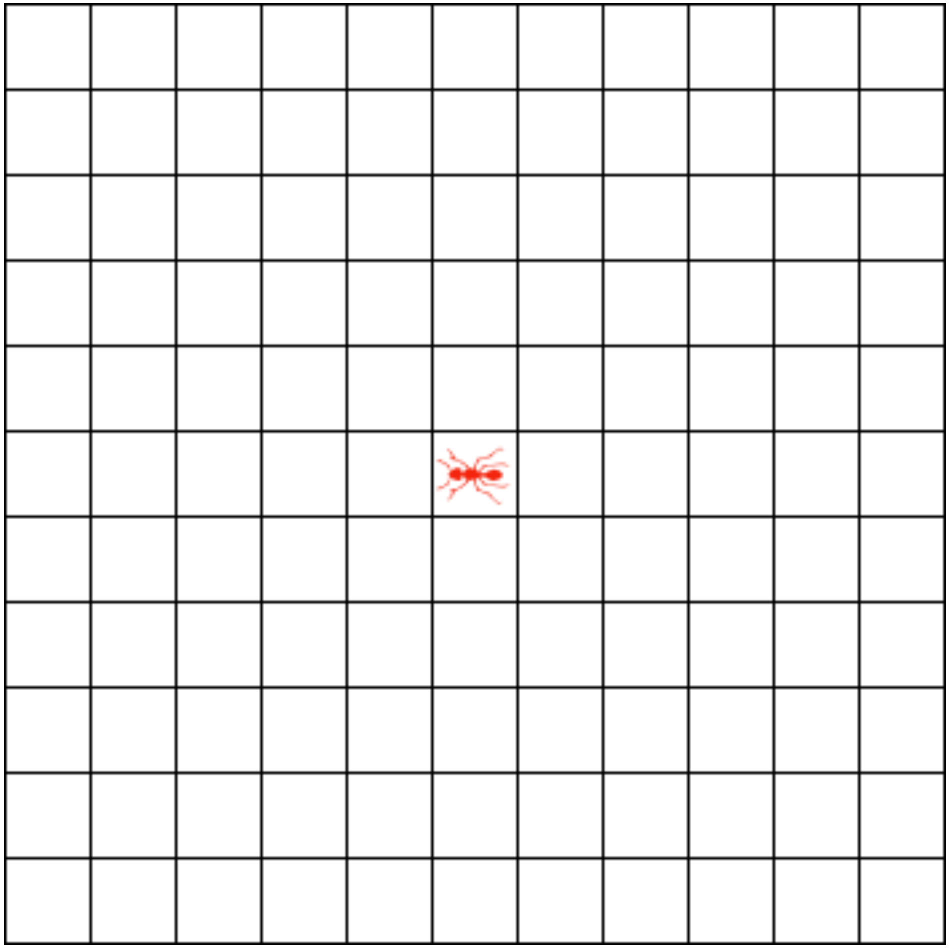
Bellos (2015) Alex Through The Looking Glass

Langton’s ant

Squares on a plane are coloured either black or white. Starting configuration - identify one square as the “ant" pointing up. The ant can travel in any of the four cardinal directions at each step it takes. The "ant" moves according to the rules below:

- At a white square, turn 90° clockwise, flip the color of the square, move forward one unit
- At a black square, turn 90° counter-clockwise, flip the color of the square, move forward one unit

Langton's ant



Animation: https://www.youtube.com/watch?v=F8-c2bawttU

Probability Theory

Formal definition: cellular automata on inf line

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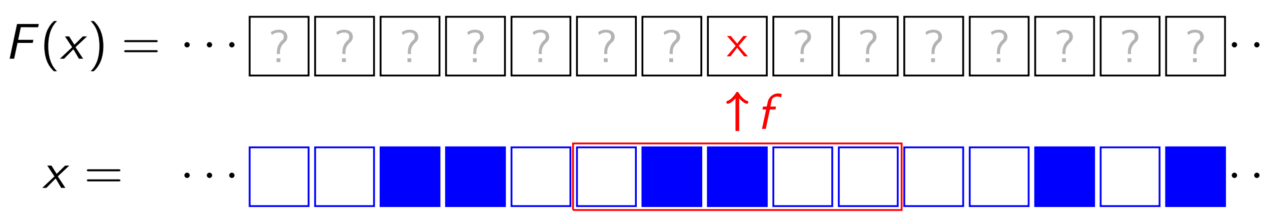

$\mathcal{A} = \{\square, \blacksquare\}$, $r = 2$

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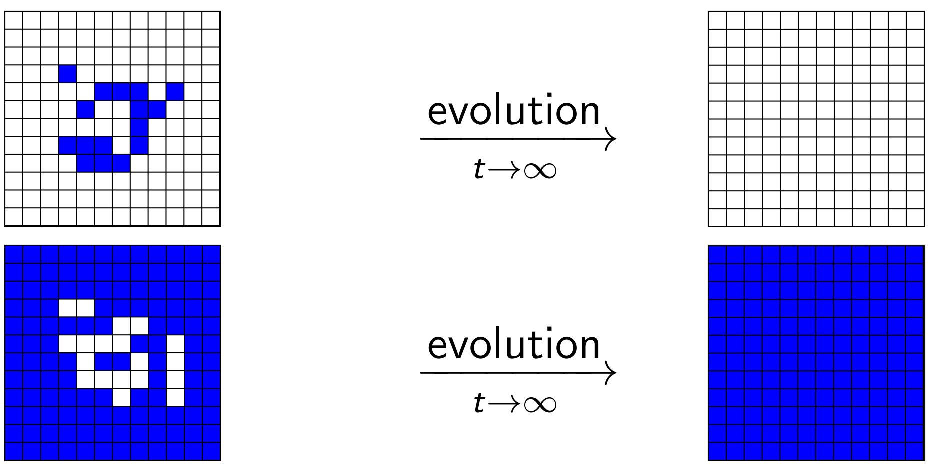
Probability Theory

Cellular Automata is an eroder if:

- from a configuration with a finite number of blue cells, it reaches the all-white configuration,
- from a configuration with a finite number of white cells, it reaches the all-blue configuration.

Goal: being able to **correct** some finite “mistakes” occuring on a monochromatic configuration.



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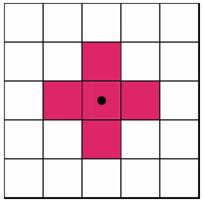
Probability Theory

Try to design a CA which is an eroder in 2D:

IDEA 1: take majority of current state and the up, down, left, right neighbours.



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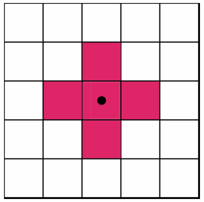
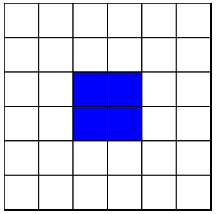


This is a fixed point!



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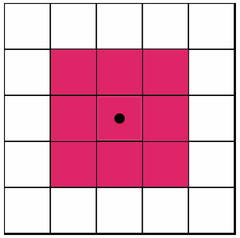
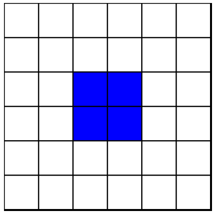
Probability Theory

Try to design a CA which is an eroder in 2D:

IDEA 2: take majority of current state and the 8 neighbouring cells (including corners).



This erodes :)



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Probability Theory

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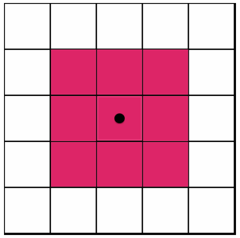
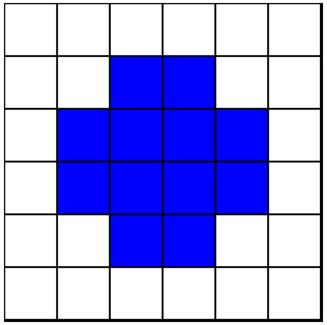


But this is a fixed point!



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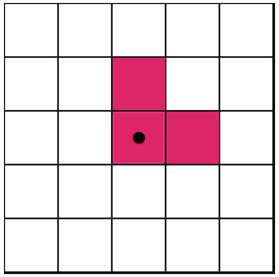
Probability Theory

Try to design a CA which is an eroder in 2D:

IDEA 3: take majority state of current state and the up, right neighbours.



This cellular automata is an eroder.

- from a configuration with a finite number of blue cells, it reaches the all-white configuration,
- from a configuration with a finite number of white cells, it reaches the all-blue configuration.

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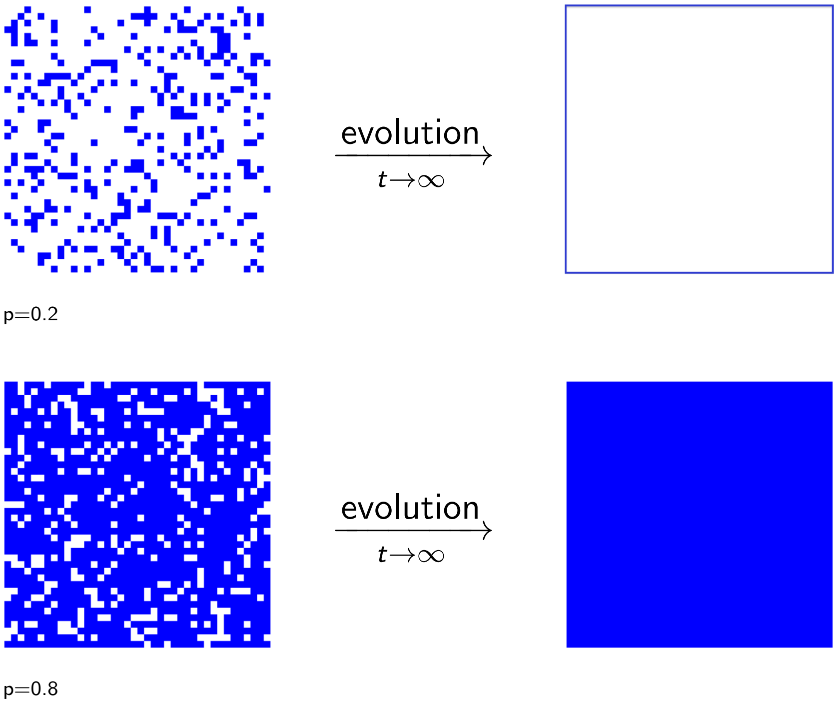
Probability Theory

Cellular Automata is a classifier if:



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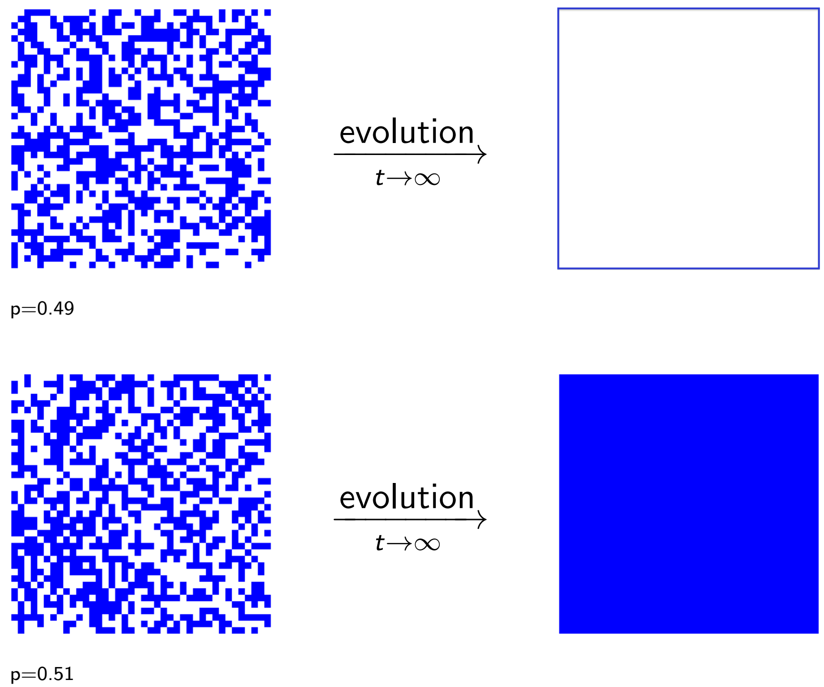
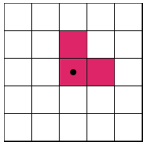
Probability Theory

Cellular Automata is a classifier if:





On infinite grid.

For fixed p<1/2 - with probability 1 evolves to all 0 state.

For fixed p>1/2 - with probability 1 evolves to all 1 state

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Probability Theory

Cellular Automata is a classifier if:

On infinite grid -

For fixed p<1/2 - with probability 1 evolves to all 0 state.

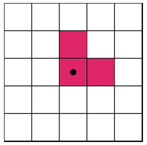
For fixed p>1/2 - with probability 1 evolves to all 1 state

Do there exist classifiers on 2D grids? **(Yes!)**

Theorem (Busic - Fates -Mairesse - Marcovici 2013)

On infinite 2D grid.

The majority CA on cell itself and up and right neighbours is a classifier.



Do there exist classifiers on 1D grids (i.e. infinite line)? **(Unknown!)**

Theorem (Taati 2015)

On infinite 1D grid. There exists a cellular automata:

For fixed p<0.0017 - with probability 1 evolves to all 0 state.

For fixed p>1-0.0017 - with probability 1 evolves to all 1 state

Open Question: Does there exist a CA which is a classifier in 1D?

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