Lecture 4: Influence and Thresholds

In our work on random graphs we have been interested in finding the thresholds for monotone sets of graphs. This has meant an analysis of the function $f(p) = \mathbb{P}(G(n,p) \in \mathcal{A})$. For non-trivial sets of graphs $\mathcal{A}$, the function satisfied $f(0) = 0$ and $f(1) = 1$ and for monotone $\mathcal{A}$ this function satisfies $f(p') \geq f(p)$ for $p' \geq p$. In this section we continue our study of this funciton $f(p)$. We will prove the Russo-Margulis lemma which allows us to calculate the derivate $\frac{d}{dp}f(p)$, i.e. the rate of change of the probability that a random graph $G_n \in \mathcal{A}$ as we change the edge probability p in $G_n \sim G(n,p)$. We will see that this derivative can be calculated in terms of what is called the *influence* of $\mathcal{A}$ which is an interesting property in its own right. .

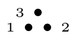
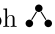
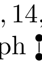
For this section we work in the general setting of a probability space over $\{0,1\}^n$.

We take the probability space Ω_n on $\mathbb{F}_2^n$ where each bit is chosen to be 1 independently with probability p (otherwise 0). For any event $\mathcal{A}_n \subset \mathbb{F}_2^n$ we write $\mu_p(\mathcal{A}_n)$ to be the probability that a randomly chosen $x \in \mathbb{F}_2^n$ lies in the set $\mathcal{A}$. We write $\mu_p(x)$ to denote the probability of the event $\mathcal{A} = \{x\}$, notice

$$\mu_p(x) = p^{\sum_i x_i}(1-p)^{n-\sum_i x_i},$$

and

$$\mu_p(\mathcal{A}_n) = \sum_x \mu_p(x).$$

Recall that any vector $x \in \mathbb{F}_2^{\binom{n}{2}}$ can be associated with a graph on n vertices: identify the $\binom{n}{2}$ positions in the vector x with the set of pairs of vertices in $[n]$ and for each co-ordinate in x, $x_e = 1$ indicates that the edge e is present in the graph. Take the convention the graph is drawn with vertex labels increasing anticlockwise starting from bottom left e.g.  and that edges listed in lexicographic order e.g. $(12, 13, 23)$ and $(12, 13, 14, 23, 24, 34)$). Now the graph  corresponds to vector $(0,1,1)$, likewise  to $(1,0,0)$ and graph  to $(0,1,1,1,1,1)$. Hence the probability space defined includes the subcase of random graphs.

In this more general context the definitions of *monotone* carries over in the way you would expect[5]. We also define monotone functions.

Definition 4.11 (monotone). A function f is *monotone*, if $f(x) \geq f(y)$ whenever $x \geq y$ (i.e. $x_i \geq y_i$ for each i). A set $\mathcal{A}_n \in \mathbb{F}_2^n$ is *monotone* if its indicator function $f_n = 1_{\mathcal{A}_n}$ is a monotone function. i.e. $f_n(x) = 1$ if $x \in \mathcal{A}$ and $f_n(x) = 0$ if $x \notin \mathcal{A}$.

In the language of voting schemes we water to say a voter has high influence if they are likely to be able to determine the outcome when we assume the rest of the popultion vote randomly. It will be on a scale of 0 to 1, where influence of 0 means they have no chance of their vote 'counting' and influence of 1 meaning that whatever the rest of the population vote the outcome would changed by the voter casting a different vote.

Definition 4.12 (pivotal). Given boolean function $f : \mathbb{F}_2^n \to \{-1, 1\}$ and $i \in [n]$ we say that i is pivotal for x if $f(x) \neq f(x \oplus i)$. For a set $A \subset \mathbb{F}_2^n$ we say i is pivotal for x if it is pivotal for its indicator function 1_A.

For the n-bit vector $x = (x_1, \ldots, x_n)$ write $x \backslash \{x_i\}$ for the $(n-1)$-bit vector $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

[5]The definition of *non-trivial* does too. A set $\mathcal{A}_n \in F_2^n$ is *non-trivial* if $\exists N$ such that $\forall n > N$, the n-vectors $\mathbf{0}$ and $\mathbf{1}$ satisfy $(0, 0, \ldots, 0) \notin \mathcal{A}_n$ and $(1, 1, \ldots, 1) \in \mathcal{A}_n$.

Definition 4.13 (influence of i-th bit, total influence). The influence of the i-th bit of a function f, is the probability that for a randomly chosen $x \backslash \{x_i\}$ changing the i-th co-ordinate of x changes f.

$$I_i^p(f) = \mu_p(\{x \ : \ x \neq f(x \oplus i)\}).$$

The influence of i-th bit of a set $\mathcal{A}$ is the influence of $f = 1_{\mathcal{A}}$. The *total influence* is the sum over all co-ordinates $I^p(f) = \sum_i I_i^p(f)$.

Notice that for a monotone set $\mathcal{A}$ the influence of the i-th bit is

$$I_i^p(\mathcal{A}) = \mu_p(\{x \ : \ (x_1, \dots, x_{i-1}, 0, x_i, \dots, x_n\} \notin \mathcal{A} \ \& \ (x_1, \dots, x_{i-1}, 1, x_i, \dots, x_n\} \in \mathcal{A}\}).$$

Example: In the parity function each co-ordinate has influence 1. For the dictator function $f = \mathrm{DICT}_1(f)$ the first co-ordinate has influence 1 the others have influence 0.

Lemma 4.14. *Let $\mathcal{A} \in F_2^n$ be a monotone event. Then*

$$\frac{d\,\mathbb{P}(\mathcal{A})}{dp} = I^p(\mathcal{A}).$$

Proof. We consider the slightly more general case where each bit x_i is chosen to be '1' independently with probability p_i, writing $I_i^{(p_1,\dots,p_n)}(A)$ for the influence of the i-th bit, i.e. the probability that the i-th bit is influential given bits $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ are chosen to be '1' independently with probabilities $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ respectively.

Hence it will suffice to show that

$$\frac{d\mathbb{P}_{(p_1,\dots,p_n)}(A)}{dp_i} = I_i^{(p_1,\dots,p_n)}p(A),$$

WLOG take $i = 1$. Now, let $X \in F_2^{n-1}$ and $Y \in F_2^{n-1}$ be defined as follows,

$$X = \{(x_2, \dots, x_n) \ : \ f(0, x_2, \dots, x_n) = 1 \text{ and } f(1, x_2, \dots, x_n) = 1\}.$$

$$Y = \{(x_2, \dots, x_n) \ : \ f(0, x_2, \dots, x_n) = 0 \text{ and } f(1, x_2, \dots, x_n) = 1\}.$$

We can express the probability of the event A in terms of X and Y,

$$\mathbb{P}_{(p_1,\dots,p_n)}(A) = \mathbb{P}_{(p_2,\dots,p_n)}(X) + \mathbb{P}_{p_1}(x_1 = 1)\mathbb{P}_{(p_2,\dots,p_n)}(Y).$$

Note that Y is the pivotal set for f, and hence $\mathbb{P}_{(p_2,\dots,p_n)}(Y)) = Inf_1(f)$ and so

$$\mathbb{P}_{(p_1,\dots,p_n)}(A) = \mathbb{P}_{(p_2,\dots,p_n)}(X) + p_1 Inf_1(f).$$

Now take the derivative of $\mathbb{P}_{(p_1,\dots,p_n)}(A)$ with respect to p_1 and we are done. □

Exercise 7. For each of the following boolean functions f, aka voting schemes, find a set S such that the function is expressible in terms of that character, i.e. $f(x) = \chi_S(x)$ or $f(x) = -\chi_S(x)$.

(a) The dictator function, $Dict_n^1(x) = x_1$.

(b) The parity function, $Par(x)$.

(c) The XOR function of the first two inputs, $f(x) = XOR(x_1, x_2)$.

(d) The constant function $f(x) = 1$.

Exercise 8. We can define an interated majority function for $n = 3^k$. The base case is $\text{Imaj}_1(x_1, x_2, x_3) = \text{Maj}_3(x_1, x_2, x_3)$ and

$$\text{Imaj}_k(x) = \text{Maj}_3(\text{Imaj}_{k-1}(x_1, \ldots, x_{3^{k-1}}), \text{Imaj}_{k-1}(x_{3^{k-1}+1}, \ldots, x_{2.3^{k-1}}), \text{IMaj}_{k-1}(x_{2.3^{k-1}+1}, \ldots, x_{3^k})).$$

For example, for $k = 2$, $\text{Imaj}_2(x_1, \ldots, x_9) = \text{Maj}_3(\text{Maj}_3(x_1, x_2, x_3), \text{Maj}_3(x_4, x_5, x_6), \text{Maj}_3(x_7, x_8, x_9))$.

(a) Calculate the influence of the i-th bit $I_i(\text{Imaj}_2)$ and total influence $I^p(\text{Imaj}_2)$.

(b) Can you calculate $I_i^p(\text{Imaj}_k)$ and $I^p(\text{Imaj}_k)$? You may take $p = 1/2$ if you like.