Lecture 5: Influence and Fourier analysis

Theorem 5.15. *For a monotone $\mathcal{A} \subset \mathbb{F}_2^n$,*

$$\frac{d}{dp}\mu_p(\mathcal{A})\Big|_{p=1/2} \leq \sqrt{n}.$$

Proof. By the Margulis-Russo result it is equivalent to show that for any monotone $\mathcal{A} \subset \mathbb{F}_2^n$ the total influence of $\mathcal{A}$ at $p = 1/2$ satisfies

$$I^{1/2}(\mathcal{A}) \leq \sqrt{n}.$$

We define a function $f : \mathbb{F}_2^n \to \{-1, 1\}$ which acts a little like the indicator for $\mathcal{A}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathcal{A} \\ -1 & \text{if } x \notin \mathcal{A}. \end{cases}$$

Notice $\mathcal{A}$ a monotone set implies that f is a monotone function. This monotonicity of f is important. For any $y \in \mathbb{F}_2^n$ and $i \in [n]$ then $f(y \oplus e_i) \neq f(y)$ if and only if

$$f(y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n) = 1 \quad \text{and} \quad f(y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n) = -1.$$

We are nearly ready to start the calculations of the proof. But first we recall the behaviour of a particular character in fourier analysis.

$$\chi_{\{i\}}(y) = (-1)^{\{i\}\cdot y} = \begin{cases} 1 & \text{if } y_i = 0 \\ -1 & \text{if } y_i = 1. \end{cases}$$

Now we can begin the calculations. The game plan is to relate $\hat{f}(\{i\})$ to the influence of f.

$$\hat{f}(\{i\}) = \frac{1}{2^n}\sum_{y\in\mathbb{F}_2^n} f(y)\chi_{\{i\}}(y) = \frac{1}{2^n}\sum_{y\in\mathbb{F}_2^n} f(y)(1_{y_i=0}(y) - 1_{y_1=1}(y)) \tag{7}$$

Notice in (7) the second equality follows by writing out $\chi_{\{i\}}(y)$ in terms of the indicator functions $1_{y_i=0}(y)$ and $1_{y_i=1}(y)$. We can now expand out the sum in (7) to get that

$$\hat{f}(\{i\}) = \frac{1}{2^n}\sum_{y\setminus\{y_i\}\in\mathbb{F}_2^{n-1}} f(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_n\} - f(y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n\}. \tag{8}$$

The equation (8) rearranges nicely. If $f(y) = f(y \oplus e_i)$ then $f(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_n\} - f(y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n\} = 0$ or if $f(y) \neq f(y \oplus e_i)$ then f monotone implies we have $f(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_n\} - f(y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n\} = -2$. The number of times this difference of two will be recorded in (8) is half the number of such y.

$$\hat{f}(\{i\}) = \frac{1}{2^n} \times (-2) \times (|\{y \ : \ f(y) = f(y \oplus e_i)\}|/2) = \frac{1}{2^n}|\{y \ : \ f(y) = f(y \oplus e_i)\}| \tag{9}$$

We have now written $\hat{f}(\{i\})$ in terms of the influence of the i-th bit. Notice that (9) calculates the probability of picking a y (under $p = 1/2$) such that $f(y) \neq f(y \oplus e_i)$. Hence,

$$\hat{f}(\{i\}) = I_i^{\frac{1}{2}}(f). \tag{10}$$

Now apply some fourier analysis. By a Corollary of Parseval (see notes from first part of course) $\sum_S \hat{f}(S)^2 = 1$. We also recall the Cauchy-Schwarz inequality for real $a_1, \ldots a_n, b_1, \ldots, b_n$ which says $(\sum_i a_i b_i)^2 \leq (\sum_i a_i^2)(\sum_i b_i^2)$ i.e. $\sum_i b_i^2 \geq (\sum_i a_i b_i)^2 / (\sum_i a_i^2)$. Then (taking $a_i = 1$ and $b_i = \hat{f}(\{i\})$ to apply Cauchy-Shwartz in the second inequality),

$$1 = \sum_S \hat{f}(S)^2 \geq \sum_{i \in [n]} \hat{f}(\{i\})^2 \geq \frac{1}{n}(\sum_i \hat{f}(\{i\}))^2 = \frac{1}{n}(\sum_i I_i^{\frac{1}{2}}(f))^2 = \frac{1}{n}(I(f))^2. \tag{11}$$

This, (11), is exactly what we want. It says $I(f) \leq \sqrt{n}$. □

Observe that the monotone condition is necessary. If we allow any possible set $\mathcal{A}$ then we could take $\mathcal{A} = \{x \ \sum_i x_i \text{ is even }\}$, i.e. so that $f = 1_{\mathcal{A}}$ is the parity function which has influence n.

We will also see that the lower bound in the theorem is right up to constatnts. In this next example we show that the majority function, which is monotone, has total influence of $\Theta(\sqrt{n})$.

For this calculation we will need stirling's formula which gives the approximate growth rate of the factorial function.

Lemma 5.16 (Stirling's formula)**.**

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right)$$

The proof of Stirling's formula is not part of this course but we use the result itself to calculate the approximate size of the middle binomial coefficient $\binom{2m}{m}$.

$$\binom{2m}{m} = \frac{\sqrt{2\pi m}}{\sqrt{2\pi m}\sqrt{2\pi m}} \frac{(2m)^{2m}}{m^m m^m}\left(1 + O\left(\frac{1}{m}\right)\right) = \frac{1}{\sqrt{2\pi m}} 2^{2m}(1 + O(1)) = \Theta\left(\frac{4^m}{\sqrt{m}}\right).$$

Example 5.17 (Influence of Majority). For odd n, denote by Maj_n the function that returns 1 if more 1's than 0's in x and -1 otherwise. We show $I^{\frac{1}{2}}(Maj_n) = \Theta(\sqrt{n})$. As all co-ordinates have the same influence it is enough to show the influence of the first co-ordinate is what we want, i.e. $I_1^{\frac{1}{2}}(Maj_n) = \Theta(\frac{1}{\sqrt{n}})$.

Recall that $x \oplus e_1$ denotes the vector x after the first co-ordinate has been swapped (e.g. $(1, 1, 0) \oplus e_1 = (0, 1, 0)$).

$$\begin{aligned} I_1^{\frac{1}{2}}(Maj_n) &= \mu_{\frac{1}{2}}(\{x \ : \ f(x \oplus e_1) \neq f(x)\}) \\ &= \frac{1}{2^n}|\{x \ : \ f(x \oplus e_1) \neq f(x)\}| \\ &= \frac{1}{2^{n-1}}\{(x_2, \ldots, x_n) \ : \ \text{exactly half the } x_i \text{ are } 1\ \}) \\ &= \frac{1}{2^{n-1}}\binom{n-1}{\frac{n-1}{2}} \end{aligned}$$

We can now substitute our bound on the middle binomial, everything cancels (except the desired square root) and we are finished the calculation. (The m above being $(n-1)/2$, note that n odd guarantees that this is an integer).

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\begin{aligned}
I_1^{\frac{1}{2}}(Maj_n) &= \frac{1}{2^{n-1}}\binom{n-1}{\frac{n-1}{2}} \\
&= \Theta\left(\frac{1}{2^{n-1}}4^{\frac{n-1}{2}}\frac{1}{\sqrt{n}}\right) \\
&= \Theta\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}
$$