Permutations in binary trees and split trees

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The definition

Subpermutations in a permutation

- Let $\sigma_1, \ldots, \sigma_n$ be a permutation of $\{1, \ldots, n\}$.
- If $i < j$ and $\sigma_i > \sigma_j$, then the pair (σ_i, σ_j) is called an *inversion*.

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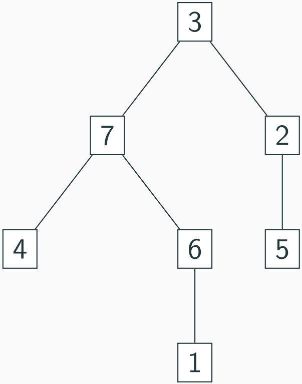
Permutations in a fixed tree

- Let T be a tree with node set V.
- Let λ be node labeling $\lambda : V \to \{1, \ldots, |V|\}$.
- For σ a permutation of $\{1, \ldots, k\}$. Let

$$R_\sigma(T, \lambda) = \sum_{u_1 < \cdots < u_k} 1_{[\lambda(u_1, \ldots, u_k) = \sigma]}.$$



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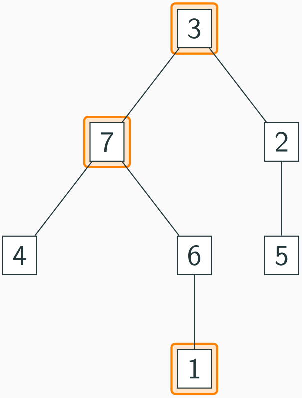

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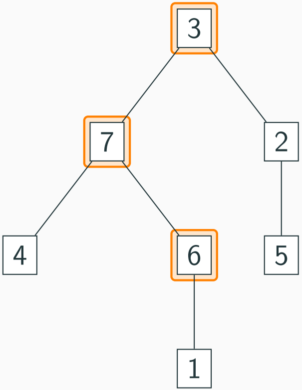


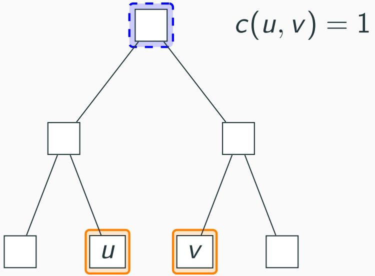
Permutations in fixed trees

k, h-total common ancestors

- For k nodes $v_1, \ldots, v_k$, let $c(v_1, \ldots, v_k)$ be the number of ancestors that they share.
- We define

$$\Upsilon_k^h(T) \stackrel{\text{def}}{=} \sum_{v_1,\ldots,v_k} c(v_1, \ldots, v_k) \prod_{i=1}^{k} \binom{d(v_i)}{h-2}.$$

- Note that $\Upsilon_k(T) = \Upsilon_k^2(T)$ and $\Upsilon_1^2(T)$ is total path length.



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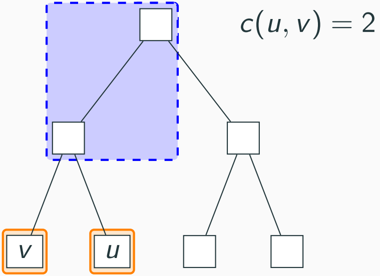


Cumulants

- The cumulant-generating function of a r.v. X is

$$K_X(t) = \log \mathbb{E}\left[e^{tX}\right].$$

- The cumulants $\varkappa_k(X)$ are defined by

$$K_X(t) = \sum_{k\geq 1} \varkappa_k(X) \frac{t^k}{k!}.$$

- We can compute centralized-moments, μ_k, from cumulants, $\{\kappa_j\}_{j\leq k}$, and vice-versa by Bell polynomials; e.g.

$$\mu_6 = \varkappa_6 + 15\kappa_4\kappa_2 + 10\kappa_3^2 + 15\kappa_2^3.$$

$$\kappa_6 = \mu_6 - 15\mu_4\mu_2 - 10\mu_3^2 + 30\mu_2^3$$

Cumulants for permutation σ $R_\sigma(T)$

Theorem 2 (Cai et al. '19)

$$\mathbb{E}\left[R_{21}(T)\right] = \varkappa_1(R_{12}(T)) = \frac{1}{2}(\Upsilon_1^2(T) - |V|),$$

More generally, for $k \geq 1$,

$$\varkappa_{2k+1}(R_{21}(T)) = 0, \qquad \varkappa_{2k}(R_{21}(T)) = \frac{B_{2k}}{2k}(\Upsilon_{2k}^2(T) - |V|),$$

where B_k denotes the k-th Bernoulli number.

Cumulants for permutation σ $R_\sigma(T)$

$\sigma = \sigma_1 \dots \sigma_k$ is a fixed permutation

T_n be the complete binary tree of depth n.

$\varkappa_r = \varkappa_r(R(\sigma, T_n))$ is r-th cumulant of $R(\sigma, T_n)$.

Theorem 1

$$\mathbb{E}\left[R_\sigma(T)\right] = \varkappa_1(R_\sigma(T)) = \frac{1}{k!}\Upsilon_1^k(T)(1+o(1)),$$

and for $r \geq 2$,

$$\varkappa_r = D_{\sigma,r}\Upsilon_r^k(T_n) + o\big(\Upsilon_r^k(T_n)\big)$$

[0] Lackner Panholzer '15: runs in randomly labelled random trees.

Cumulants for permutation σ $R_\sigma(T)$

$\sigma = \sigma_1 \ldots \sigma_k$ is a fixed permutation
T_n be the complete binary tree of depth n.

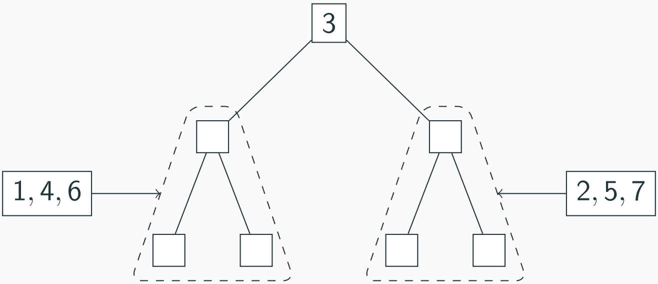
Corollary For permutations σ of length 3,

$$\mathbb{V}(R(\sigma, T_n)) = \begin{cases} \frac{1}{45}\Upsilon_2^3(T_n)(1+o(1)) & \text{for } \sigma = 123, 132, 312, 321 \\ \frac{1}{180}\Upsilon_2^3(T_n)(1+o(1)) & \text{for } \sigma = 213, 231 \end{cases}$$

[0] Lackner Panholzer '15: runs in randomly labelled random trees.

A key observation for inversions

- Let Z_o be the number of inversions involving the root.
- Then Z_o and the numbers of inversions in the left subtree and right subtree are independent.
- Proof by conditioning on the labels that go the left and the right.



Inversions key lemma

- Let z_v be the size of the subtree at v.
- Let Z_v be the number of inversions involving v and one of its descendants.

Lemma 1

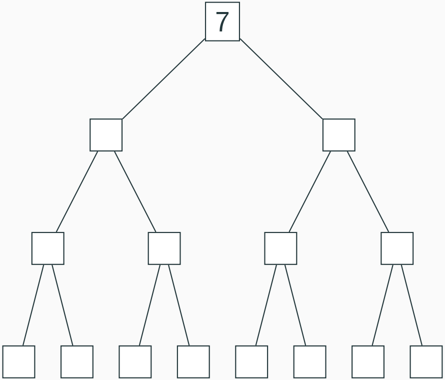
Let T be a fixed tree. Then

$$R_{21}(T) \stackrel{\mathrm{d}}{=} \sum_{v \in V} Z_v,$$

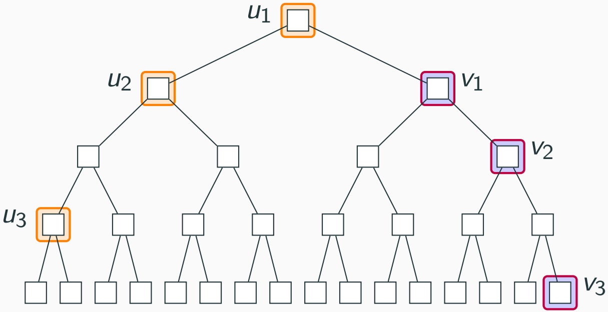
where $\{Z_v\}_{v \in V}$ are independent random variables, and $Z_v \sim \mathrm{Unif}\{0, 1, \ldots, z_v - 1\}$.

Inversions key observation

- Let Z_o be the number of 213 involving the root as top vertex.
- Let Z_u be the number of 213 involving the left child of the root as top vertex.
- Then Z_o and Z_u not independent.



$$\mathbb{V}(R_{321}(T,\lambda))$$
$$= \Big(\sum_{u_1<u_2<u_3} 1_{[\lambda(u_1,u_2,u_3)=321]} - \tfrac{1}{6}\Big)\Big(\sum_{v_1<v_2<v_3} 1_{[\lambda(v_1,v_2,v_3)=321]} - \tfrac{1}{6}\Big)$$



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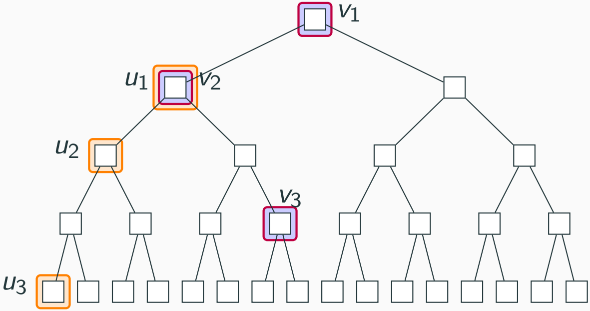

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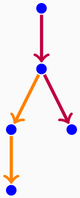

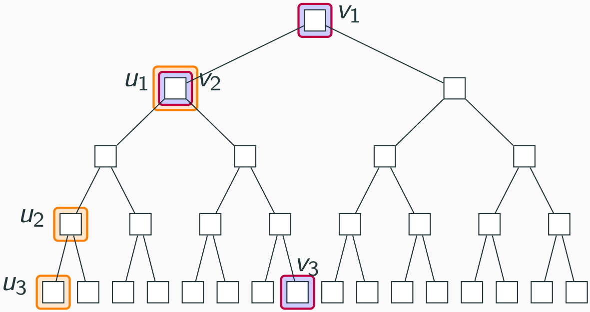

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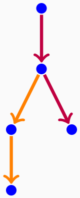

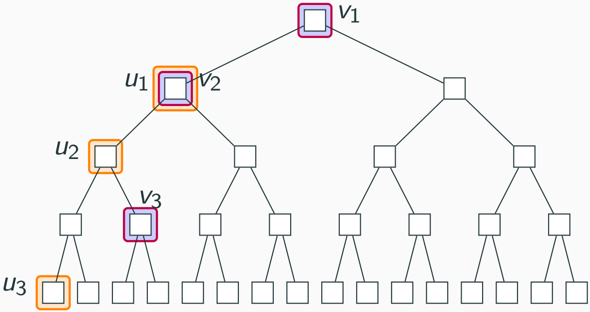

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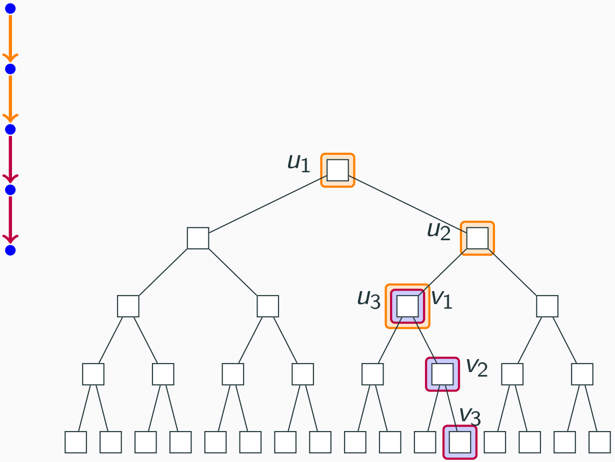

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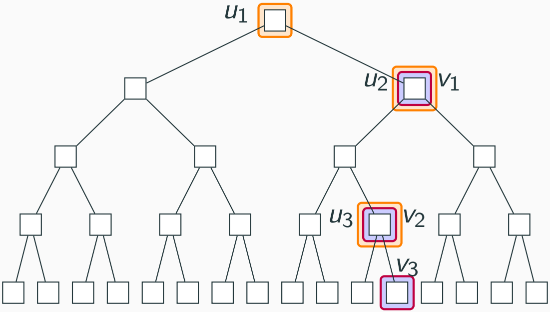

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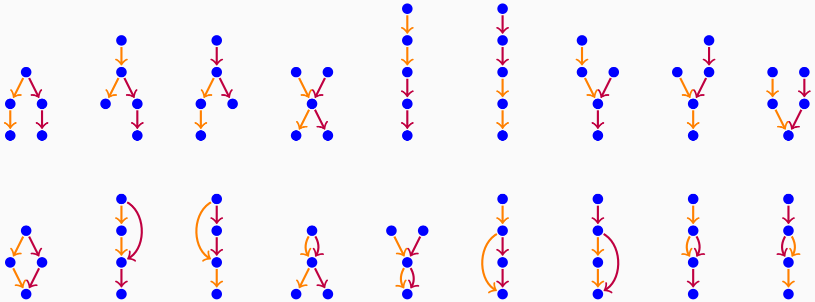
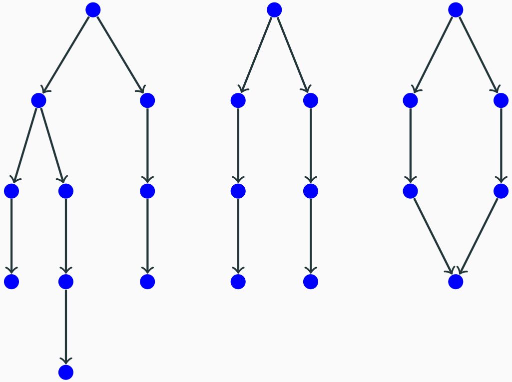
$$= \sum_{\vec{H}} c_{\vec{H}} \cdot \ \#\text{times } \vec{H} \text{ embeds into tree}$$



Method of moments

$$\mathbb{E}\Big[\Big(R_{2431}(T,\lambda)\Big)^7\Big]$$



Embeddings of digraphs into trees

Embeddings of digraphs into trees

$\vec{H}$ constant size digraph

T_n sequence of trees on growing number of vertices.

$[\vec{H}]_{T_n} \stackrel{\text{def}}{=} |\{\iota : V(\vec{H}) \to V(T_n) \text{ s.t. if } u < v \text{ in } \vec{H} \text{ then } \iota(u) < \iota(v) \text{ in } T_n\}|$

Let P_ℓ be the rooted path on ℓ nodes.

Then []$_{P_4} = 2$ and in general []$_{P_\ell} = 2\binom{\ell}{4}$.

Key lemma for permutations on binary trees

a_0, a_1 parameters of digraph $\vec{H}$.

Lemma Let T_n be the complete binary tree with n vertices and $\vec{H}$ be a fixed directed acyclic graph. Then

$$[\vec{H}]_{T_n} = \Theta(n^{a_0}(\ln n)^{a_1}).$$

Key lemma for permutations on binary trees

a_0, a_1 parameters of digraph $\vec{H}$.

$a_0 = \#\bullet$ - sink vertices
$a_1 = \#\bullet$ - ancestor of exactly one sink vertex
$a_2 = \#\bullet$ - ancestor of more than one sink vertex

Lemma Let T_n be the complete binary tree with n vertices and $\vec{H}$ be a fixed directed acyclic graph. Then

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Embeddings of digraphs into trees

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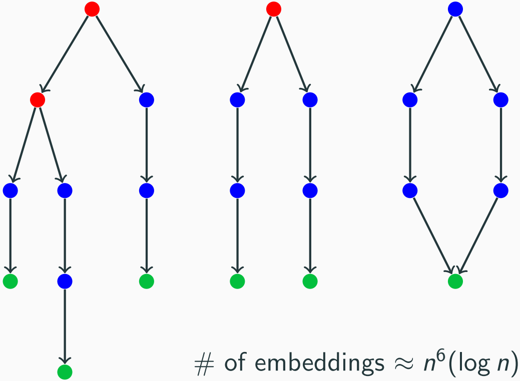
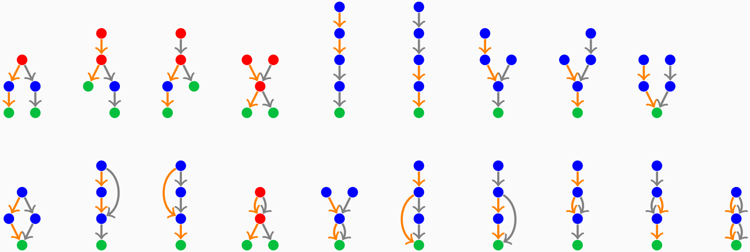


$\#$ of embeddings $\approx n^6(\log n)^{14}$

$$\mathbb{V}(R_{231}(T,\lambda))$$

$$= \Big(\sum_{u_1<u_2<u_3} 1_{[\lambda(u_1,u_2,u_3)=231]} - \tfrac{1}{6} \Big) \Big(\sum_{v_1<v_2<v_3} 1_{[\lambda(v_1,\ldots,v_3)=231]} - \tfrac{1}{6} \Big)$$

$= \sum_{\vec{H}} c_{\vec{H}} \cdot [\vec{H}]_{T_n} = c' \cdot [$  $]_{T_n} + o([$  $]_{T_n})$



Key lemma applied

$$\mathbb{E}((R_{231} - \mathbb{E}[R_{231}])^6) = (1 + o(1)) \sum_{\vec{H}} c_{\vec{H}} \cdot [\vec{H}]_{T_n}$$

Key lemma applied

$$\mathbb{E}((R_{231} - \mathbb{E}[R_{231}])^6) \quad = \quad (1 + o(1)) \sum_{\vec{H}} c_{\vec{H}} \cdot [\vec{H}]_{T_n}$$

$\mu_6 = \varkappa_6 + 15\kappa_4\kappa_2 + 10\kappa_3^2 + 15\kappa_2^3$

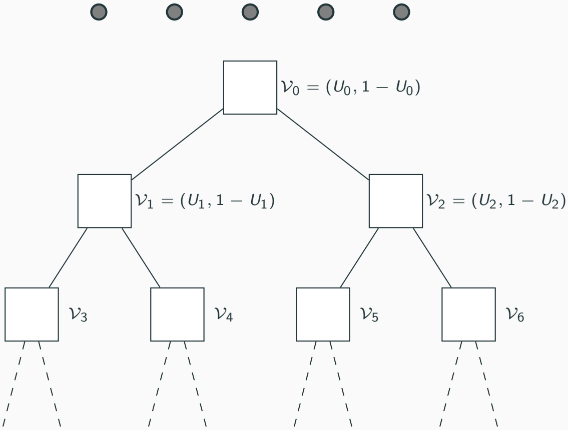
$\kappa_2 = c[S_2]_{T_n}$

$\kappa_3 = c'[S_3]_{T_n}$

$\kappa_4 = c''[S_4]_{T_n}$

Split trees

Binary search tree as a split tree



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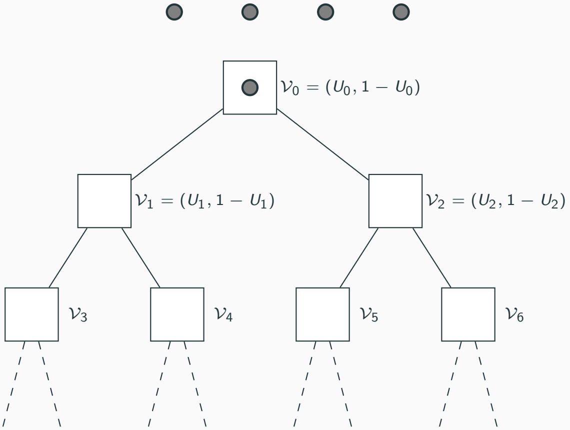

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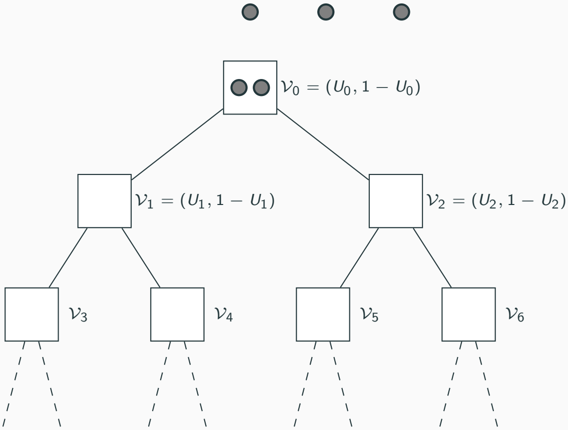

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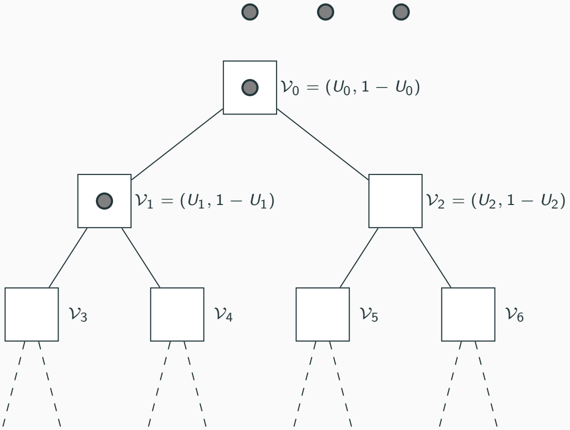

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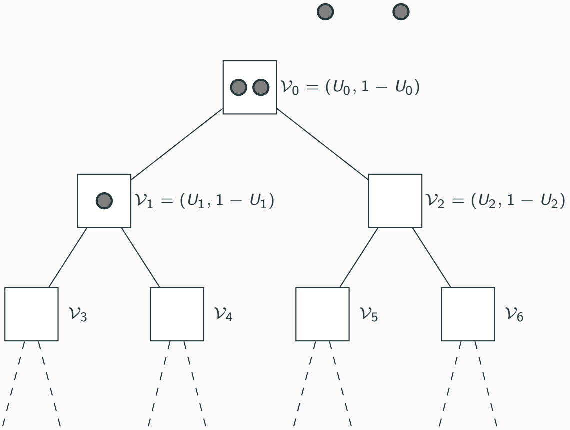

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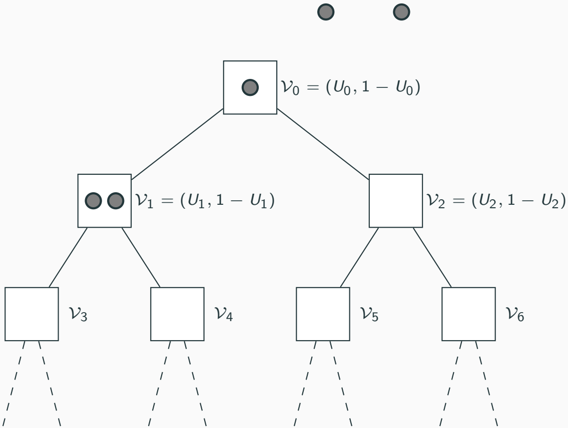

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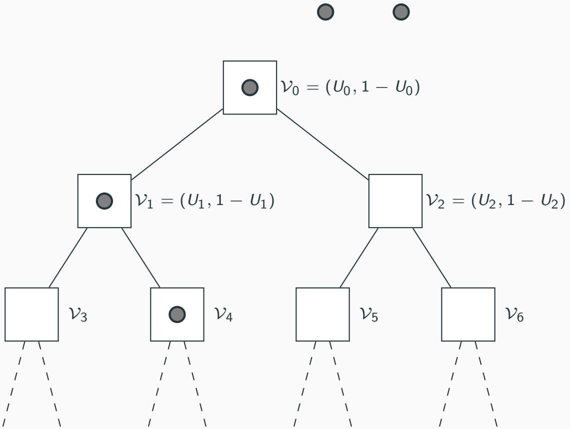

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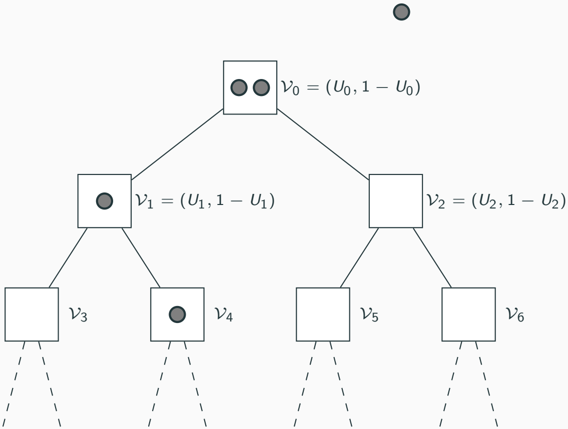

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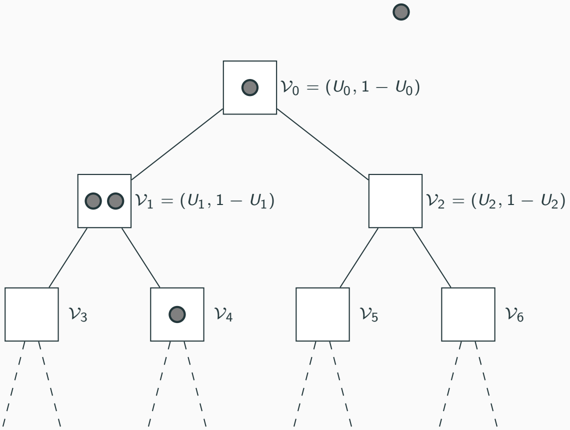

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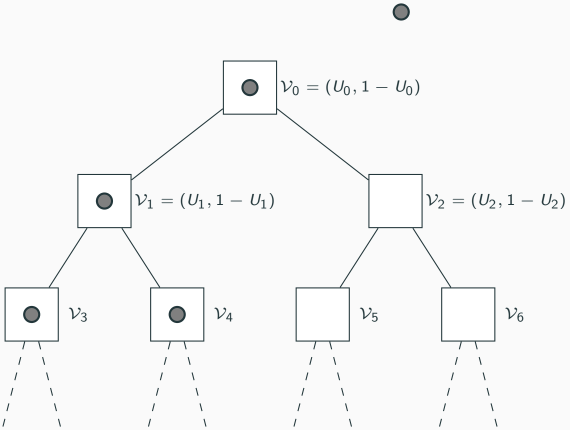

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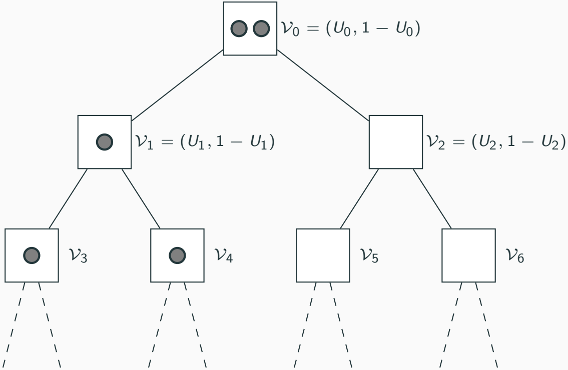

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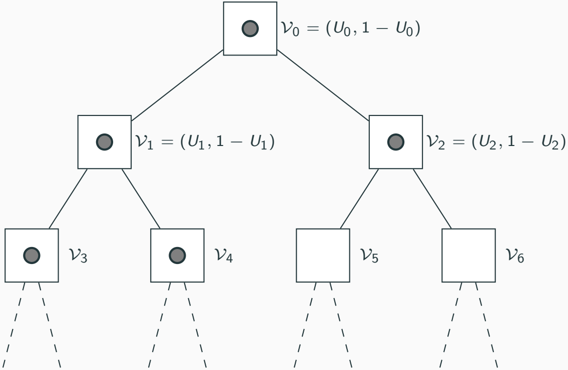

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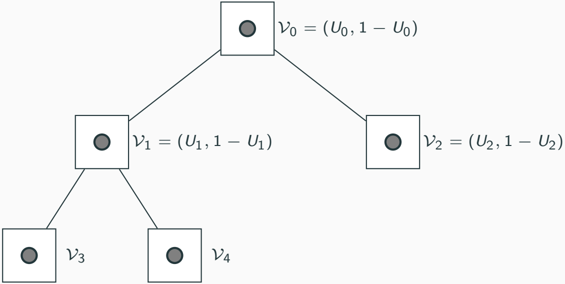


Binary search tree as a split tree

- Begin with infinite binary tree.
- Each node is a "bucket" of capacity one.
- Each node given split vector $\mathcal{V} = (U, 1 - U)$ chosen independently.
- n balls start at root one by one.
- When a bucket overflows, the extra goes to child nodes chosen at random according to $\mathcal{V}$.
- All empty buckets are removed in the end.
- Parameters: "bucket capacity", "# kept", "# guaranted booty", split vector distribution.

[0] Devroye '99

Cumulants for permutation σ $R_\sigma(T_n)$

$\sigma = \sigma_1 \dots \sigma_k$ is a fixed permutation

T_n be the split tree with n balls and $s_0 > 0$.

$\varkappa_r = \varkappa_r(R(\sigma, T_n))$ is r-th cumulant of $R(\sigma, T_n)$.

Theorem 2
Whp the following holds

$$\mathbb{E}\left[R_\sigma(T_n)\right] = \varkappa_1(R_\sigma(T)) = \frac{1}{k!}\Upsilon_1^k(T_n)(1 + o(1)),$$

and for $r \geq 2$,

$$\varkappa_r = D_{\sigma,r}\Upsilon_r^k(T_n) + o\big(\Upsilon_r^k(T_n)\big)$$

Key lemma for permutations on split trees

$a_0 = \#\bullet$ - sink vertices

$a_1 = \#\bullet$ - ancestor of exactly one sink vertex

$a_2 = \#\bullet$ - ancestor of more than one sink vertex

Lemma Let T_n be a split tree with n balls and $\vec{H}$ be a fixed directed acyclic graph. Then whp

$$[\vec{H}]_{T_n} = \Omega(n^{a_0}(\ln n)^{a_1}).$$

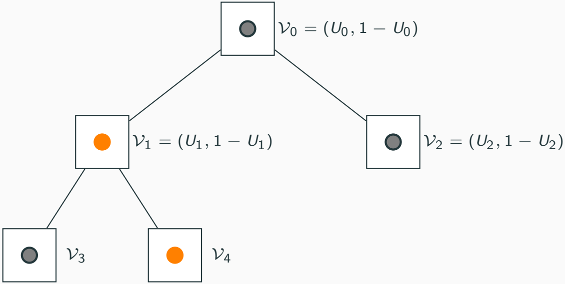
$$\mathbb{E}[\vec{H}]_{T_n} = \Theta(n^{a_0}(\ln n)^{a_1}).$$

$a_0 = \#\bullet$ - sink vertices

$a_1 = \#\bullet$ - ancestor of exactly one sink vertex

$a_2 = \#\bullet$ - ancestor of more than one sink vertex

$$\mathbb{E}[\vec{H}]_{T_n} = O(n^{a_0}(\ln n)^{a_1}).$$



$$\sum_{b,b'} \mathbb{E} \ \#\text{nodes above both balls } b, b' \leq C' n^2.$$

$a_0 = \#\bullet$ - sink vertices

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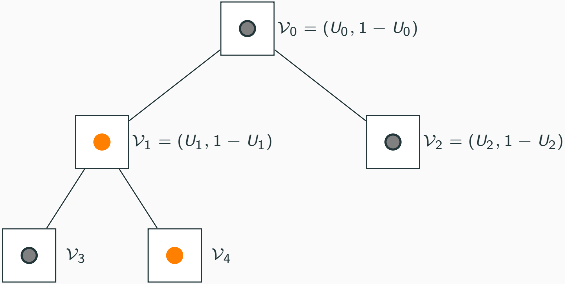
$$\mathbb{E}[\vec{H}]_{T_n} = O(n^{a_0}(\ln n)^{a_1}).$$



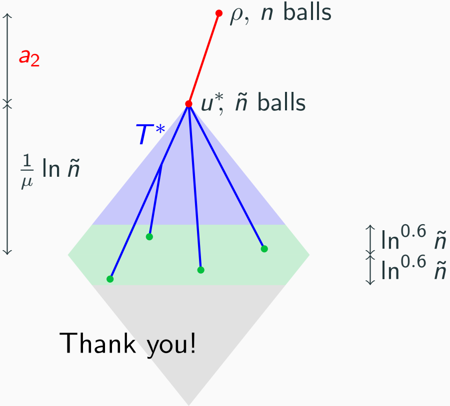
$$\sum_{b,b'} \mathbb{E} \ \#\text{nodes above both balls } b, b' \leq C' n^2.$$

$a_0 = \#$● - sink vertices

$a_1 = \#$● - ancestor of exactly one sink vertex

$a_2 = \#$● - ancestor of more than one sink vertex

$$\text{whp } [\vec{H}]_{T_n} = \Omega(n^{a_0} (\ln n)^{a_1}).$$



Thank you!

Open: The constants $D_{\alpha,r}$

$\|\alpha\|$	$\alpha_1 \in ?$	1	2	3	4	5
2	$\{1,2\}$	$\frac{1}{2}$	$\frac{1}{2^2\cdot 3}$	0	$\frac{-1}{2^3\cdot 3\cdot 5}$	0
3	$\{1,3\}$	$\frac{1}{2\cdot 3}$	$\frac{1}{3^2\cdot 5}$	$\frac{2}{3^3\cdot 5\cdot 7}$	$\frac{-2}{3^3\cdot 5^2\cdot 7}$	$\frac{-2^3}{3^4\cdot 5\cdot 7\cdot 11}$
3	$\{2\}$	$\frac{1}{2\cdot 3}$	$\frac{1}{2^2\cdot 3^2\cdot 5}$	$\frac{-1}{2^2\cdot 3^3\cdot 5\cdot 7}$	$\frac{-1}{2^3\cdot 3^3\cdot 5^2\cdot 7}$	$\frac{1}{2^2\cdot 3^4\cdot 5\cdot 7\cdot 11}$
4	$\{1,4\}$	$\frac{1}{2^3\cdot 3}$	$\frac{1}{2^6\cdot 7}$	$\frac{1}{2^8\cdot 5\cdot 7}$	$\frac{-3}{2^{11}\cdot 5\cdot 7^2\cdot 13}$	$\frac{-3}{2^{12}\cdot 7^2\cdot 13}$
4	$\{2,3\}$	$\frac{1}{2^3\cdot 3}$	$\frac{13}{2^6\cdot 3^2\cdot 5\cdot 7}$	$\frac{-1}{2^8\cdot 3^3\cdot 5\cdot 7}$	$\frac{-5591}{2^{11}\cdot 3^3\cdot 5^2\cdot 7^2\cdot 11\cdot 13}$	$\frac{199}{2^{12}\cdot 3^4\cdot 5\cdot 7^2\cdot 11\cdot 13}$
5	$\{1,5\}$	$\frac{1}{2^3\cdot 3\cdot 5}$	$\frac{1}{2^2\cdot 3^4\cdot 5^2}$	$\frac{1}{2^2\cdot 3^4\cdot 5^3\cdot 13}$	$\frac{29}{2^3\cdot 3^7\cdot 5^4\cdot 13\cdot 17}$	$\frac{-107}{2^2\cdot 3^8\cdot 5^5\cdot 7\cdot 13\cdot 17}$
5	$\{2,4\}$	$\frac{1}{2^3\cdot 3\cdot 5}$	$\frac{37}{2^6\cdot 3^4\cdot 5^2\cdot 7}$	$\frac{53}{2^8\cdot 3^4\cdot 5^3\cdot 7\cdot 11\cdot 13}$	$\frac{-849839}{2^{11}\cdot 3^7\cdot 5^4\cdot 7^2\cdot 11\cdot 13\cdot 17}$	$\frac{-1041109}{2^{12}\cdot 3^8\cdot 5^5\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19}$
5	$\{3\}$	$\frac{1}{2^3\cdot 3\cdot 5}$	$\frac{1}{2^6\cdot 3\cdot 5^2\cdot 7}$	$\frac{-19}{2^8\cdot 3^3\cdot 5^3\cdot 7\cdot 11\cdot 13}$	$\frac{-73^2}{2^{11}\cdot 3^3\cdot 5^4\cdot 7^2\cdot 11\cdot 13\cdot 17}$	$\frac{10061}{2^{12}\cdot 3^4\cdot 5^5\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19}$
6	$\{1,6\}$	$\frac{1}{2^4\cdot 3^2\cdot 5}$	$\frac{1}{2^8\cdot 3^4\cdot 11}$	$\frac{1}{2^{13}\cdot 3^6\cdot 11}$	$\frac{1}{2^{14}\cdot 3^7\cdot 7\cdot 11^2}$	$\frac{-19}{2^{19}\cdot 3^9\cdot 7\cdot 11^2\cdot 13}$
6	$\{2,5\}$	$\frac{1}{2^4\cdot 3^2\cdot 5}$	$\frac{1}{2^8\cdot 3^2\cdot 5^2\cdot 11}$	$\frac{509}{2^{13}\cdot 3^6\cdot 5^3\cdot 7\cdot 11\cdot 13}$	$\frac{-233\cdot 619}{2^{13}\cdot 3^7\cdot 5^4\cdot 7\cdot 11^2\cdot 13\cdot 17\cdot 19}$	$\frac{-18928549}{2^{19}\cdot 3^9\cdot 5^5\cdot 7\cdot 11^2\cdot 13\cdot 17\cdot 19\cdot 23}$
6	$\{3,5\}$	$\frac{1}{2^4\cdot 3^2\cdot 5}$	$\frac{43}{2^8\cdot 3^4\cdot 5^2\cdot 7\cdot 11}$	$\frac{1}{2^{11}\cdot 3^6\cdot 5^3\cdot 7\cdot 13}$	$\frac{-211\cdot 9341}{2^{15}\cdot 3^7\cdot 5^4\cdot 7^2\cdot 11^2\cdot 13\cdot 17\cdot 19}$	$\frac{-47\cdot 3701}{2^{17}\cdot 3^9\cdot 5^5\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23}$

Figure 8: A table showing values of $D_{\alpha,r}$ for α of lengths 2 to 6 and moments $r = 1,\ldots,5$.

Open: The constants $D_{\alpha,r}$

- $P(r)$ partiions of $[r]$ e.g. $\{1\}\{2,3\} \in P(3)$.
- $D_{\alpha,r} = \sum_{\tau \in P(r)} (-1)^{|\tau|-1}(|\tau|-1)! \prod_{s \in \tau} \beta(|s|, r, \alpha)$
- $\beta(|s|, r, \alpha) = \frac{\left((\alpha_1-1)|s|\right)!((k-\alpha_1)|s|)!}{\left((\alpha_1-1)!(k-\alpha_1)!\right)^{|s|}\left((k-1)|s|+1\right)!}$

Open: show directly?

$$\sum_{\tau \in P(r)} (-1)^{|\tau|-1}(|\tau|-1)! \prod_{s \in \tau} \frac{1}{|s|+1} = (-1)^r \frac{B_r}{r}$$

Open: Galton Watson trees

Let $e(t)$ Brownian excursion and
$\eta \stackrel{\mathrm{def}}{=} 4\int_{0\le s\le t\le 1} \min_{s\le u\le t} e(u)\mathrm{d}s\mathrm{d}t$.

Theorem 3 (Cai et al. '19)

Assume that $\mathbb{E}[\xi]=1$, $\mathrm{Var}(\xi)=\sigma^2\in(0,\infty)$, *and* $\mathbb{E}\left[e^{\alpha\xi}\right]<\infty$ *for some* $\alpha>0$. *Then*

$$\frac{I(T_n)-\frac{1}{2}\Upsilon(T_n)}{n^{5/4}} \xrightarrow{\mathrm{d}} \frac{1}{\sqrt{12\sigma}}\sqrt{\eta}\ N(0,1).$$

Let $X_n=(I(T_n)-\mathbb{E}[I(T_n)])/n^{3/2}$, then

$$X_n=\frac{I(T_n)-\frac{1}{2}\Upsilon(T_n)}{n^{3/2}}+\frac{\Upsilon(T_n)-\mathbb{E}[\Upsilon(T_n)]}{2n^{3/2}}.$$

[0] Aldous '91, Panholzer-Seitz '12

Open: Galton Watson trees

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Split trees: $[\vec{H}]_{T_n} = \Omega(n^{a_0}(\ln n)^{a_1})$.

Galton-Watson: $[\vec{H}]_{T_n} = \tilde{\Omega}(n^{a_0}(\sqrt{n})^{a_1})$? or $\tilde{\Omega}(n^{a_0}(\sqrt{n})^{a_1+a_2})$?.